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Massless spectra of three generation $\mathrm{U}(N)$ heterotic string vacua

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Abstract: We provide the methods to compute the complete massless spectra of a class of recently introduced supersymmetric $E_8 \times E_8$ heterotic string models which invoke vector bundles with $\mathrm{U}(N)$ structure group on simply connected Calabi-Yau manifolds and which yield flipped SU(5) and MSSM string vacua of potential phenomenological interest. We apply Leray spectral sequences in order to derive the localisation of the cohomology groups $H^i(X, V_a \otimes V_b)$, $H^i(X, \bigwedge^2 V)$ and $H^i(X, \mathbf{S}^2 V)$ for vector bundles defined via Fourier-Mukai transforms on elliptically fibered Calabi-Yau manifolds. By the method of bundle extensions we define a stable $\mathrm{U}(4)$ vector bundle leading to the first flipped SU(5) model with just three generations, i.e. without any vector-like matter. Along the way, we propose the notion of λ-stability for heterotic bundles.

Keywords: Superstring Vacua, Superstrings and Heterotic Strings.

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Contents

1. Introduction

Since its discovery in 1985 the heterotic string [1] has been considered as a promising candidate to yield four-dimensional string vacua whose low-energy effective action resemble the Standard Model (SM) of particle physics or an extension thereof. Different constructions, based on the subsequent discovery of D-branes, such as intersecting D-brane models provide an alternative way to realize many of the Standard Model properties in concrete four-dimensional string vacua.[1] Whereas these latter constructions are well-suited to directly yield the Standard Model gauge symmetry at the string scale, for GUT-like theories the $E_8 \times E_8$ heterotic string seems to be particularly natural.

[1]For references see e.g. the latest reviews [2] or [3].

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In the last couple of years there has been a revival of attempts to construct realistic $E_8 \times E_8$ string vacua on Calabi-Yau manifolds. In fact using advanced techniques for the construction of vector bundles, models have been found with the MSSM massless charged particle spectrum. Some of them are completely supersymmetric [4, 5] whereas others include an explicit supersymmetry breaking hidden E_8 or M-theoretic bulk sector [6, 7].

The philosophy of these constructions, pioneered in [8], is to embed an SU(4) or SU(5) bundle into one E_8 which first gives rise to an SO(10) and SU(5) observable GUT model, respectively. Due to the absence of candidate GUT Higgs fields, this gauge symmetry has to be broken by discrete Wilson lines. In fact, most of the effort has gone into the investigation of Calabi-Yau manifolds admitting non-trivial discrete Wilson lines and the construction of appropriate equivariant vector bundles [9 – 11].

Based on the methods developed in [12, 13], an alternative to this procedure has recently been presented in [14]. There it is shown that by allowing also vector bundles with $\mathrm{U}(N)$ structure groups, the massless spectrum can contain GUT Higgs scalars. This approach therefore circumvents the necessity of working on manifolds with non-trivial fundamental group and opens up the way for heterotic model building on much more general background manifolds. In particular, the technology for the construction of stable vector bundles on elliptically fibered Calabi-Yau manifolds [15 – 17] can directly be employed.

Concretely, the approach of [14] provides two alternative strategies. The first option consists of embedding a vector bundle with structure group $\mathrm{SU}(4) \times \mathrm{U}(1)$ into the first E_8. This engineers the observable gauge symmetry $\mathrm{SU}(5) \times \mathrm{U}(1)$. Under the $\mathrm{U}(1)$ the prospective SM particles carry exactly the charge known from $\mathrm{U}(1)_X$ in the flipped SU(5) GUT scenario [18, 19]. However, without further refinements the $\mathrm{U}(1)$ becomes massive due to the Green-Schwarz mechanism. To remedy this one can embed in addition a line bundle into the second E_8 factor yielding an observable $E_7 \times \mathrm{U}(1)$ gauge symmetry. Under certain conditions on the vector bundles one linear combination of the two $\mathrm{U}(1)$ factors from the first and the second E_8 remains massless, eventually giving rise to a supersymmetric flipped SU(5) GUT model. The role of the GUT Higgs pair is played by the component in the $[\mathbf{10} - \overline{\mathbf{10}}]$ neutral under the SM gauge group. One physical motivation to study the resulting models are the known phenomenologically attractive field-theoretic features of the flipped SU(5) scenario [20, 21].[2] These include a high degree of proton stability, among others due to a natural solution to the doublet-triplet splitting problem, and distinguish the flipped from the non-flipped SU(5) models. Whereas in the purely field-theoretic flipped SU(5) model the GUT scale value of the $\mathrm{U}(1)_X$ gauge coupling is in principle a free parameter, in the stringy flipped SU(5) of [14] the three tree-level gauge couplings are uniquely determined and do not unify at the string or GUT scale. Nonetheless gauge coupling unification can in principle be achieved by a suitable tuning of the stringy threshold corrections. Moreover, there appear, in general, exotic massless states, which turn out to be all vector-like as soon as one requires that the $\mathrm{U}(1)_X$ stays massless. It is important to note that the presence of these exotics is by no means a definite prediction of this string theoretic realisation of flipped SU(5) since they can well be avoided by a suitable choice

[2] Recent alternative constructions of flipped SU(5) vacua in various contexts include [22 – 26].

of bundle data. In fact, it is the main result of the present paper to exemplify that this is indeed possible.

The second option studied in [14] is to embed an $\mathrm{SU}(5)\times \mathrm{U}(1)$ bundle into one E_8 and a second line bundle into the other E_8. This yields string vacua with just the Standard Model gauge symmetry and, again, only very few non-chiral exotic matter states, which may or may not be present depending on the details of the compactification data. In [14], it was also carried out a successful computer search for chiral three-generation flipped SU(5) and direct MSSM models on elliptically fibered Calabi-Yau manifolds, where the base was allowed to be either a Hirzebruch surface or a del Pezzo surface with $r \leq 4$.

The final aim of this paper is to continue the model search of [14] and to demonstrate the existence of string vacua of the two types described above and with as little vector-like exotic matter as possible. These could then serve as the starting point for concrete phenomenological studies. Technically, our models will be based on the extension of spectral cover bundles with structure group $\mathrm{U}(N)$ on elliptically fibered Calabi-Yau manifolds, as pioneered in [15 − 17]. The computation of the vector-like matter spectrum of these $\mathrm{U}(N)$ bundles requires some technology from algebraic geometry which may be slightly beyond the everyday needs in the physics literature. Before addressing the construction of string vacua of the above type in section 4, we therefore have to spend some time diving into the details of the spectral cover construction. In particular, it will be necessary to generalise the methods developed in [27].

For self-consistency of this paper we will begin section 2 by reviewing the construction of μ-stable bundles over elliptically fibered Calabi-Yau manifolds via the method of spectral covers. We will also explain the method of bundle extensions, which allows one to construct stable bundles of higher from lower rank ones. We then propose a new notion of stability, which should be relevant for vector bundles for heterotic strings and which includes, similarly to Π-stability for D-branes, higher perturbative and non-perturbative corrections. We call bundles which are stable in this sense Λ-stable. Finally, we recall the criterion stated in [28] for stability of extensions of spectral cover bundles and prove it in appendix A.

In section 3 we will partly review and partly newly derive the main technical tools for the computation of the various cohomology groups relevant for determining the massless modes for the string compactifications of interest. By applying the Leray spectral sequence, we will first recall that the cohomology of line bundles over the Calabi-Yau manifold can be computed from line bundles over the base manifold of the elliptic fibration. Moreover, we will explicitly show that the cohomology of the tensor product of two $\mathrm{U}(N)$ bundles localises on the intersection curve of the two spectral covers and can be determined by computing solely the cohomology of a certain line bundle over this support curve. This is in agreement with the special case considered in [27] that one of the bundles is a trivial line. Therefore, eventually the entire computation is transformed into computing the cohomology of line bundles over curves given by complete intersections of two surfaces. These can be evaluated using Koszul sequences. With these results available we move forward and newly compute the cohomologies of the bundles $\bigwedge^2 V$ and $\mathbf{S}^2 V$. Note that the formula we derive differs from the one found in [27].

Equipped with these powerful mathematical results, in section 4 we address the construction of flipped SU(5) heterotic string vacua by using vector bundles with structure group SU(4) × U(1). After recalling the main ingredients of [14] we provide a new globally consistent supersymmetric three generation example, for which the U(4) bundle is defined as a stable extension of two U(2) bundles. The model exhibits precisely one pair of GUT Higgs fields as required for GUT breaking down to the MSSM gauge group. The particle spectrum of the resulting $SU(3) \times SU(2) \times U(1)_Y$ vacuum is precisely that of the supersymmetric Standard Model spectrum with no extra vector-like matter but a number of additional electro-weak Higgs pairs. To the best of our knowledge, this is the first consistent flipped SU(5) string model with these properties in the literature.

2. Stable $\mathrm{U}(n)$ bundles via spectral covers

In sections 2.1–2.3 we review the construction of μ-stable $\mathrm{U}(n)$ vector bundles over elliptically fibered Calabi-Yau manifolds via the spectral cover method [15, 17, 16]. More information can also be found e.g. in [29]. We then recall in section 2.4 the definition of vector bundles as non-trivial extensions of such spectral cover bundles. These parts are meant as a pedagogical introduction to this topic in order to make the present article accessible to the non-expert reader and may safely be skipped by specialists. Section 2.5 analyses the stability concept appropriate for our applications, relegating a proof of the stability of our bundles to the appendix A.

2.1 Elliptically fibered Calabi-Yau manifolds

An elliptically fibered complex threefold X is given by a complex two-surface B, the base space, together with an analytic map

$$\pi : X \to B, \tag{2.1}$$

where fibers over each point b in the base

$$E_b = \pi^{-1}(b) \tag{2.2}$$

are elliptic curves. Recall that an elliptic curve is a two-torus with a complex structure inducing an abelian group law. In particular it contains a distinguished point p acting as the zero element in this group.

We require the fibration X to admit a global section $\sigma : B \to X$, assigning to every point in the base $b \in B$ the zero element $\sigma(b) = p \in E_b$ on the fiber. This section embeds the base as a sub-manifold into X and we will often not distinguish between B as a complex two-fold and σ as its image in X. The associated homology class in $H_4(X, \mathbb{Z})$ then intersects the fibre class precisely once. It will be useful to introduce also the class in $H^2(X, \mathbb{Z})$ Poincaré dual to the class of σ. In slight abuse of notation, it will also be referred to as σ. The respective meaning will hopefully always be clear from the context. Its cohomological self-intersection can be proven to be [15]

$$\sigma \cdot \sigma = -\sigma \cdot \pi^* c_1(B). \tag{2.3}$$

Likewise, we introduce $F \in H^4(X, \mathbb{Z})$ as the Poincaré dual to the fibre class. The fact that the base class intersects the class of the generic fibre once is reflected in the cohomological intersection form

$$\sigma \cdot F = 1. \tag{2.4}$$

This shows that F is actually the Hodge dual to the two-form σ. Now that we are at it, we state for later purposes the simple fact that the intersection form of the pull-back to X of two classes α and β in $H^2(B, \mathbb{Z})$ is given by the pull-back of the intersection on B,

$$\pi^*(\alpha) \cdot \pi^*(\beta) = \pi^*(\alpha \cdot \beta) = (\alpha \cdot \beta)\, F. \tag{2.5}$$

Let us now turn our attention to the elliptic fibre. Elliptic curves can be described as the hyperplane in $\mathbb{CP}^2$ defined by the homogeneous Weierstrass equation

$$zy^2 = 4x^3 - g_2 x z^2 - g_3 z^3, \tag{2.6}$$

where x, y, z are homogeneous coordinates on $\mathbb{CP}^2$ and g_2 and g_3 define the complex structure. When we define a family of elliptic curves over the base, x, y, z and g_2 and g_3 must be promoted to global sections of certain powers of some line bundle $\mathcal{L}$ on B. The choice of this line bundle and the global sections x, y, z will define the fibration.

In order to promote equation (2.6) to a vanishing condition of a global section of a line bundle on B, we choose x, y, z to be sections of $\mathcal{L}^2$, $\mathcal{L}^3$ and $\mathcal{O}$ whereas g_2 and g_3 appear as sections of $\mathcal{L}^4$ and $\mathcal{L}^6$, respectively. If the fibration X is to be Calabi-Yau, the first Chern class of the tangent bundle T must vanish,

$$c_1(X) = 0. \tag{2.7}$$

As shown e.g. in [29], this implies $\mathcal{L} = K_B^{-1}$, where K_B is the canonical bundle of the base space. It follows that K_B^{-4} and K_B^{-6} must have sections g_2 and g_3, respectively. The surfaces compatible with this condition are found to be del Pezzo, Hirzebruch, Enriques and blow-ups of Hirzebruch surfaces [30]. Note, however, that the construction of stable holomorphic bundles on elliptically fibered three-folds does not hinge upon the Calabi-Yau property. In order to simplify the mathematical apparatus, we nonetheless assume (2.7) in the sequel.

Friedman-Morgen-Witten (FMW) showed that on such spaces the Chern classes of the tangent bundle of the total space follow from the Chern classes of the base space. In particular, the second Chern class of the tangent bundle can be computed as

$$c_2(X) = 12\sigma \cdot \pi^* c_1(B) + \left(11 c_1(B)^2 + c_2(B)\right) F. \tag{2.8}$$

For later purposes let us recall that on X there exists a holomorphic involution τ acting solely on the fiber as $\tau : y \to -y$. The fixed point locus of τ consists of two components. The first component is given by $x = z = 0$ and arbitrary y, which is nothing else than the section σ. The second component is defined by $y = 0$ and is therefore a triple cover of

B, whose homology class was derived in [15] as $3\sigma + 3c_1(B)$. The homology class of the complete fixed point surface is therefore

$$\sigma_\tau = 4\sigma + 3c_1(B), \tag{2.9}$$

where the factor 4σ reflects the four fixed points of the holomorphic involution (-1) on T^2.

2.2 The spectral cover construction

The basic idea of the spectral cover method is to first construct a μ-stable $\mathrm{U}(n)$ or $\mathrm{SU}(n)$ bundle on the elliptic fibre over each point of the base, which is then extended over the whole manifold by gluing the data together suitably. Recall that in general, a $\mathrm{U}(n)$ or $\mathrm{SU}(n)$ bundle defines a rank n complex vector bundle. Such a rank n bundle over an elliptic curve must, in order to satisfy the Hermitian Yang-Mills equation, be of degree zero. More precisely, it can be shown to be isomorphic to the direct sum of n complex line bundles

$$\mathcal{V}|_{E_b} = \mathcal{N}_1 \oplus \ldots \oplus \mathcal{N}_n, \tag{2.10}$$

each of which has to be of zero degree. If $G = \mathrm{SU}(n)$ as opposed to $\mathrm{U}(n)$, $\mathcal{V}|_{E_b}$ must in addition be of trivial determinant, i.e. $\bigotimes_{i=1}^n \mathcal{N}_i = \mathcal{O}_{E_b}$. The zero degree condition on $\mathcal{N}_i$ implies that there exists for each $\mathcal{N}_i$ a meromorphic section with precisely one zero at some Q_i and a pole at p, i.e. $\mathcal{N}_i = \mathcal{O}_{E_b}(Q_i - p)$. Consequently, stable $(S)\,\mathrm{U}(n)$ bundles on an elliptic curve are in one-to-one correspondence with the unordered n-tuple of points Q_i, and the reduction of $\mathrm{U}(N)$ to $\mathrm{SU}(n)$ is encoded in the additional requirement that $\sum_i (Q_i - p) = 0$ in the group law of the elliptic curve.

Having understood the restriction of a rank n bundle $\mathcal{V}$ to each elliptic fibre, we can now proceed to constructing the whole of $\mathcal{V}$. In intuitive terms, the above implies that over an elliptically fibered manifold a $\mathrm{U}(n)$ vector bundle determines a set of n points, varying over the base. More precisely, the bundle $\mathcal{V}$ over X with the property that for a generic fiber E_b

$$\mathcal{V}|_{E_b} = \bigoplus_{i=1}^{n} \mathcal{O}(Q_i - p) \tag{2.11}$$

uniquely defines an n-fold cover C of B, the spectral cover. It is defined by a projection

$$\pi_C : C \to B \qquad \text{and} \qquad C \cap E_b = \pi_C^{-1}(b) = \bigcup_i Q_i. \tag{2.12}$$

C, which is a hypersurface in X, can be conveniently described as the vanishing locus of some global section of the line bundle $\mathcal{O}_X(n\sigma + \pi^*\eta)$. Here η denotes some effective class in $H^2(B,\mathbb{Z})$. In particular, this implies that the Poincaré dual two-form of C is in

$$[C] = n\sigma + \pi^*\eta \in H^2(X,\mathbb{Z}) \tag{2.13}$$

Note that under the involution τ the class $[C]$ is invariant, while the spectral cover C is in general not invariant.

Several distinct bundles over X may well give rise to the same spectral cover C since the latter only encodes the information about the restriction of $\mathcal{V}$ to the fibre E_b. To recover $\mathcal{V}$ from the spectral data we need to specify in addition how it varies over the base, i.e. $\mathcal{V}|_B$. As discussed in [15] this is uniquely accomplished by the so-called spectral line bundle $\mathcal{N}$ on C with the property

$$\pi_{C*}\mathcal{N} = \mathcal{V}|_B. \tag{2.14}$$

We can formalise these results by introducing the notion of the Poincaré line bundle $\mathcal{P}$. For this purpose, consider the fibre product $X \times_B X'$ as the set of pairs $(z_1, z_2) \in X \times X'$ with $\pi(z_1) = \pi(z_2)$. Furthermore we need to introduce π_1 and π_2 as the projections on the first and second factor, respectively. Moreover, σ_1 denotes the section $\sigma_1 : B \to X \to X \times_B X'$ and σ_2 the section $\sigma_2 : B \to X' \to X \times_B X'$. Then $\mathcal{P}$ is defined as the bundle over $X \times_B X'$ with the two properties

$$\mathcal{P}|_{E_b \times x} \simeq \mathcal{P}|_{x \times E_b} \simeq \mathcal{O}_{E_b}(x - p), \qquad \mathcal{P}|_{\sigma_i} = \mathcal{O}_{\sigma_i},\ i = 1, 2. \tag{2.15}$$

Introducing the diagonal divisor Δ, the first Chern class of the Poincaré line bundle is

$$c_1(\mathcal{P}) = \Delta - \sigma_1 - \sigma_2 - c_1(B). \tag{2.16}$$

Note that Δ satisfies the relations

$$\Delta^2 = -\Delta \cdot c_1(B), \quad \Delta \cdot \sigma_i = \sigma_1 \cdot \sigma_2. \tag{2.17}$$

We will denote by $\mathcal{P}_B$ the restriction of $\mathcal{P}$ to $X \times_B C$. Now by definition, $\pi_{1*}(\mathcal{P}_B)|_x = \bigoplus_i \mathcal{O}_{E_{\pi(x)}}(Q_i - p)$, as is clear from the fact that $C \cap E_b = \bigcup_i Q_i$ and the first property in (2.15). This remains true if we tensor $\mathcal{P}_B$ with $\pi_2^*(\mathcal{N})$ for some line bundle $\mathcal{N}$ on C. After all, $\pi_2^*(\mathcal{N})$ as a bundle on X is trivial when restricted to the fibre E_b. On the other hand, $\mathcal{P}|_{\sigma \times_B X'}$ is likewise trivial due to the second property in (2.15), and so $(\pi_{1*}(\pi_2^*\mathcal{N} \otimes \mathcal{P_B}))|_B = \pi_{1*}(\pi_2^*\mathcal{N} \otimes \mathcal{P_B})|_{\sigma_2})$ is simply given by $\pi_{C*}\mathcal{N}$. In other words, the bundle

$$\mathcal{V} = \pi_{1*}(\pi_2^*\mathcal{N} \otimes \mathcal{P}_B) \tag{2.18}$$

indeed exhibits the two defining properties (2.11) and (2.14). This establishes the definition of an $(S)\,\mathrm{U}(n)$ bundle on the elliptically fibered Calabi-Yau threefold in terms of the spectral data $(C, \mathcal{N})$. We reiterate that we will only consider the case that the restriction of the bundle to the elliptic fibre is an $\mathrm{SU}(n)$ bundle, i.e. that C is as in (2.13).

The bundles constructed so far are μ-semi-stable on a generic elliptic fiber. It has been shown in [17], Theorem 7.1, that an irreducible spectral cover is a sufficient condition in order to obtain a μ-stable vector bundle.[3] There are two simple conditions on the curve η [27] which ensure the existence of an irreducible spectral cover:

[3] In fact, the proof guarantees stability of the bundle with respect to an ample class, i.e. a Kähler class, $J = \epsilon\sigma + J_B$ such that the Kähler parameter of the fiber lies in a certain range near the boundary of the Kähler cone, that is for sufficiently small ϵ. Since the value of ϵ is not known, in all models involving the spectral cover constructions it is therefore a subtle issue if the region of stability overlaps with the perturbative regime, which is needed to have control over non-perturbative effects. In all examples which will be relevant for us, the constraints will leave us enough freedom to go to regions of the Kähler cone where ϵ is much smaller than J_B.

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- The linear system $|\eta|$ is base-point free.
- The class $\eta - nc_1(B)$ is effective.

We will be more specific about their implications when it comes to a discussion of the properties of the base.

We now give the topological invariants of the bundle $\mathcal{V}$ defined by (2.18). The working horse for this computation is the Grothendieck-Riemann-Roch (GRR) theorem. Applying this theorem to the projection $\pi_1 : X \times_B C \to X$ allows us to compute the Chern classes of $\mathcal{V}$

$$\pi_{1*}\left(e^{c_1(\mathcal{N}\otimes\mathcal{P}_B)}\,\mathrm{Td}(X \times_B C)\right) = \mathrm{ch}(\mathcal{V})\mathrm{Td}(X). \tag{2.19}$$

As discussed in [15], this relates in particular $c_1(\mathcal{N})$ and $c_1(\mathcal{V})$ as

$$c_1(\mathcal{N}) = \frac{1}{n}\,\pi_C^* c_1(\mathcal{V})|_B - \frac{1}{2}\,c_1(C) + \frac{1}{2}\,\pi_C^* c_1(B) + \gamma \tag{2.20}$$

in terms of the cohomology class γ satisfying

$$\pi_{C*}\gamma = 0. \tag{2.21}$$

One can prove that γ can in general be written as

$$\gamma = \lambda(n\sigma - \pi_C^*\eta + n\pi_C^* c_1(B)), \tag{2.22}$$

where $\lambda \in \mathbb{Q}$. Note furthermore that $c_1(C)$ is minus the first Chern class of the canonical bundle $K_C = \mathcal{O}(C)$ on C, i.e. $c_1(C) = -n\sigma - \pi_C^*\eta$. We now parameterise $c_1(\mathcal{V})$ by some element $c_1(\zeta) \in H^2(B,\mathbb{Z})$ to be specified momentarily,

$$c_1(\mathcal{V}) = \pi^* c_1(\zeta). \tag{2.23}$$

Putting everything together, we have

$$c_1(\mathcal{N}) = n\left(\tfrac{1}{2} + \lambda\right)\sigma + \left(\tfrac{1}{2} - \lambda\right)\pi_C^*\eta + \left(\tfrac{1}{2} + n\lambda\right)\pi_C^* c_1(B) + \tfrac{1}{n}\,\pi_C^* c_1(\zeta). \tag{2.24}$$

Since $c_1(\mathcal{N})$ and $c_1(\mathcal{V})$ must be an integer class, not every value of $\lambda \in \mathbb{Q}$ and $c_1(\zeta) \in H^2(B,\mathbb{Z})$ is allowed in the ansatz for $c_1(\mathcal{V})$. Rather they are subject to the constraints

$$\begin{gathered} n\left(\tfrac{1}{2} + \lambda\right) \in \mathbb{Z}, \\ \left(\tfrac{1}{2} - \lambda\right)\eta + \left(n\lambda + \tfrac{1}{2}\right)c_1(B) + \tfrac{1}{n}\,c_1(\zeta) \in H^2(B,\mathbb{Z}), \end{gathered} \tag{2.25}$$

but can otherwise be chosen arbitrarily. Note that if we are interested in $\mathrm{SU}(n)$ bundles as e.g. in [15], then simply $c_1(\zeta) = 0$ so that $c_1(\mathcal{V}) = 0$. All other consistent choices yield $\mathrm{U}(n)$ bundles. Allowing non-trivial values for $c_1(\mathcal{V})$ was first considered in [31] and motivated by the relative Fourier-Mukai transform, but we will not invoke this picture here.[4] Further

[4] To recover their expressions, simply set $c_1(\zeta) = \eta_E - \frac{n}{2} c_1(B)$ in the notation of [31].

applications of the GRR theorem lead to the following expressions for the second and third Chern classes [15, 32, 31]

$$\begin{aligned}\mathrm{ch}_2(\mathcal{V}) &= -\sigma\cdot\pi^*\eta + \left(\tfrac{1}{2n}c_1(\zeta)^2 - \omega\right)F,\\ \mathrm{ch}_3(\mathcal{V}) &= \lambda\eta\cdot(\eta - nc_1(B)) - \tfrac{1}{n}\,c_1(\zeta)\cdot\eta,\end{aligned} \tag{2.26}$$

where

$$\omega = -\frac{1}{24}c_1(B)^2(n^3-n) + \frac{1}{2}\left(\lambda^2 - \frac{1}{4}\right)n\eta\cdot(\eta - nc_1(B)). \tag{2.27}$$

Note that $\mathrm{ch}_3(V)$ has already been integrated over the fiber.

As we emphasized several times, this kind of construction only gives bundles with trivial first Chern class as restricted to the elliptic fibres. To be more general, we can however twist the bundle $\mathcal{V}$ defined via the spectral cover construction with an additional line bundle $\mathcal{Q}$ on X with [33]

$$c_1(\mathcal{Q}) = q\sigma + \pi^*(c_1(\zeta_Q)), \tag{2.28}$$

where $\pi^*(c_1(\zeta_Q)) \in H^2(X,\mathbb{Z})$. The resulting $\mathrm{U}(n)$ bundle

$$V = \mathcal{V}\otimes\mathcal{Q} \tag{2.29}$$

is μ-stable precisely if the original bundle $\mathcal{V}$ is. The Chern classes for V are straightforwardly computed from the ones of $\mathcal{V}$ and from $c_1(\mathcal{Q})$. Note that the contribution form $\pi^*(c_1(\zeta_Q))$ can always be absorbed into an additive shift of $c_1(\zeta)$ by $nc_1(\zeta_Q)$. We will not make use of $\mathrm{U}(n)$ bundles with $q \neq 0$ in this article. The above Chern characters are therefore sufficient for our purposes.

2.3 Del Pezzo surfaces

As alluded to already, the Calabi-Yau condition imposes severe constraints on which complex two-surfaces are eligible as base manifolds of our elliptic fibration. Among the possibilities classified in [30] we can choose as the base manifold one of the del Pezzo surfaces dP_r with $r = 0,\ldots,9$. The surface dP_r is defined by blowing up r points in generic position on $\mathbb{P}_2$. This means that $H^2(\mathrm{dP_r},\mathbb{Z})$ is generated by the $r+1$ elements $l, E_1,\ldots,E_r$, where l is the hyperplane class inherited from $\mathbb{P}_2$ and the E_m denote the r exceptional cycles introduced by the blow-ups. The intersection form can be computed as

$$l\cdot l = 1,\quad l\cdot E_m = 0,\quad E_m\cdot E_n = -\delta_{m,n}. \tag{2.30}$$

The first equation follows from the fact that two representatives of the class l define two complex lines in generic position which clearly intersect precisely once. The self-intersection for the blow-ups is the usual one for exceptional cycles. Furthermore, a complex line in generic position does not pass through any of the blown-ups, thus $l\cdot E_m = 0$.

The Chern classes read

$$c_1(dP_r) = 3l - \sum_{m=1}^{r} E_m, \qquad c_2(dP_r) = 3 + r. \tag{2.31}$$

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r	Generators	#
1	$E_1,\, l - E_1$	2
2	$E_i,\, l - E_1 - E_2$	3
3	$E_i,\, l - E_i - E_j$	6
4	$E_i,\, l - E_i - E_j$	10
5	$E_i,\, l - E_i - E_j,\, 2l - E_1 - E_2 - E_3 - E_4 - E_5$	16
6	$E_i,\, l - E_i - E_j,\, 2l - E_i - E_j - E_k - E_l - E_m$	27
7	$E_i,\, l - E_i - E_j,\, 2l - E_i - E_j - E_k - E_l - E_m,$ $3l - 2E_i - E_j - E_k - E_l - E_m - E_n - E_o$	56
8	$E_i,\, l - E_i - E_j,\, 2l - E_i - E_j - E_k - E_l - E_m,$ $3l - 2E_i - E_j - E_k - E_l - E_m - E_n - E_o,$ $4l - 2(E_i + E_j + E_k) - \sum_{i=1}^{5} E_{m_i},$ $5l - 2\sum_{i=1}^{6} E_{m_i} - E_k - E_l,\, 6l - 3E_i - 2\sum_{i=1}^{7} E_{m_i}$	240
9	$f = 3 - \sum_{i=1}^{9} E_i$, and $\{y_a\}$ with $y_a^2 = -1,\, y_a \cdot f = 1$	∞

Table 1: Generators for the Mori cone of each dP_r, $r = 1, \ldots, 9$. All indices $i, j, \ldots \in \{1, \ldots, r\}$ in the table are distinct. The effective classes can be written as linear combinations of the generators with integer non-negative coefficients.

We recognize the part involving l as the first Chern class of the parent $\mathbb{P}_2$. For the second Chern class of the elliptic threefold X we obtain, applying (2.8),

$$c_2(X) = 12\sigma c_1(B) + (102 - 10r)\, F. \tag{2.32}$$

Now for a vector bundle V_i we can expand η_i and $c_1(\zeta_i)$ in a cohomological basis

$$\begin{aligned} \eta_i &= \eta_i^{(0)}\, l + \sum_{m=1}^{r} \eta_i^{(m)}\, E_m \equiv (\eta_i^{(0)}, \eta_i^{(1)}, \ldots, \eta_i^{(r)}) \\ c_1(\zeta_i) &= \zeta_i^{(0)}\, l + \sum_{m=1}^{r} \zeta_i^{(m)}\, E_m \equiv (\zeta_i^{(0)}, \zeta_i^{(1)}, \ldots, \zeta_i^{(r)}). \end{aligned} \tag{2.33}$$

As mentioned before we have to require for stability that $|\eta|$ is effective and that $\eta - n\, c_1(B)$ is effective. Fortunately, the generating system for the cone of effective curves on dP_r has been given in [34] and we list the reformulated result of [27] in table 1 for completeness. Recall that a general effective class can be expanded into a linear combination of these Mori cone generators with non-negative integer coefficients.

Moreover, $|\eta|$ is known to be base point free if $\eta \cdot E \geq 0$ for every curve E with $E^2 = -1$ and $E \cdot c_1(B) = 1$. Such curves are precisely given by the generators of the Mori cone listed in table 1.

2.4 More bundles from extensions

The $\mathrm{U}(n)$ bundles constructed in the previous section can serve as the building block for a more general construction of vector bundles known as the extension method. Physically, the idea is to start with the direct sum of two bundles, $V_1 \oplus V_2$ and deform it into a new, stable bundle V. More abstractly, if such a deformation is possible, the resulting bundle V, the extension of V_2 by V_1, fits into the short exact sequence

$$0 \to V_1 \to V \to V_2 \to 0. \tag{2.34}$$

The possible deformations of $V_1 \oplus V_2$ which still fit into the exact sequence (2.34) are classified by the extension group $\mathrm{Ext}^*_X(V_2, V_1)$. In our case, since V_1 and V_2 are vector bundles and not merely coherent sheaves, the extension group is actually given by the cohomology groups $H^*(X, V_1 \otimes V_2^\vee)$. The above extension V can be chosen non-split, i.e. V is a proper deformation of $V_1 \oplus V_2$, precisely if

$$H^1(X, V_1 \otimes V_2^\vee) \neq 0. \tag{2.35}$$

The total Chern character of the extension bundle V follows from the ones of V_1 and V_2 as

$$\mathrm{ch}(V) = \mathrm{ch}(V_1) + \mathrm{ch}(V_2). \tag{2.36}$$

The cohomology groups of V, $H^*(X, V)$, can in principle be computed from $H^*(X, V_1)$ and $H^*(X, V_2)$ by exploiting the standard fact that a short exact sequence induces a long exact sequence in cohomology.

For later use we note furthermore that the exact sequence (2.34) remains exact upon tensoring each element appearing in it by a line bundle L, i.e. the sequence

$$0 \to V_1 \otimes L \to V \otimes L \to V_2 \otimes L \to 0 \tag{2.37}$$

is exact precisely if (2.34) is. This will allow us to obtain the cohomology groups of $V \otimes L$ from $H^*(X, V_i \otimes L)$ by invoking the long exact sequence in cohomology induced by (2.37).

2.5 Comments on μ- and Λ-stability

At string tree level, for a heterotic compactification to preserve supersymmetry, the vector bundle must be holomorphic and its field strength has to satisfy the Hermitian Yang-Mills (HYM) equation $g^{a\bar{b}} F_{a\bar{b}} = 0$. The latter is most conveniently rewritten in its dual version

$$\star_6 \left[J \wedge J \wedge F_i^{ab} \right] = 0, \tag{2.38}$$

where a, b are gauge indices and $i = 1, 2$ distinguishes the two E_8 factors. A solution to this equation exists if the vector bundle is $\mu-$stable and obeys the Donaldson-Uhlenbeck-Yau (DUY) condition

$$\int_X J \wedge J \wedge c_1(V) = 0. \tag{2.39}$$

Recall that a vector bundle V is called μ-stable if each subsheaf $\mathcal{F}$ of rank smaller than the rank of V satisfies $\mu(\mathcal{F}) < \mu(V)$, where the μ-slope $\mu(\mathcal{F})$ for a sheaf $\mathcal{F}$ with respect to the Kähler form J of the manifold X is defined as

$$\mu(\mathcal{F}) = \frac{1}{\mathrm{rk}\mathcal{F}} \int_X J \wedge J \wedge c_1(\mathcal{F}). \tag{2.40}$$

As has been shown in [12] by analysing the D-term supersymmetry conditions in the effective four-dimensional $\mathcal{N} = 1$ supergravity, for $\mathrm{U}(N)$ bundles there exists a one-loop correction to the DUY equation. Following the same logic which has lead to the DUY theorem, it is natural to conjecture that this is due to a corresponding stringy one-loop correction to the HYM equation of the form

$$\begin{aligned} \star_6 \Bigg[J \wedge J \wedge F_i^{ab} - \frac{\ell_s^4}{4(2\pi)^2} e^{2\phi_{10}} F_i^{ab} \wedge \left(\mathrm{tr}_{E_{8i}}(F_i \wedge F_i) - \frac{1}{2}\mathrm{tr}(R \wedge R) \right) \\ + \ell_s^4 e^{2\phi_{10}} \sum_a N_a \left(\frac{1}{2} \mp \lambda_a \right)^2 F_i^{ab} \wedge \overline{\gamma}_a \Bigg] + (\text{n.p. terms}) = 0\,. \end{aligned} \tag{2.41}$$

Here $\overline{\gamma}_a$ denotes the Poincaré dual four-form of the two-cycles wrapped by five-branes which may or may not be present in the concrete vacuum under consideration. The positions of the five-branes are parametrized by $-1/2 \leq \lambda_a \leq 1/2$ and the minus sign in the last term in (2.41) is for the first E_8 and the plus sign for the second. More information can be found in [14]. Non-renormalisation theorems for D-terms in supersymmetric theories imply that there are no higher loop corrections, but as indicated there can be non-perturbative ones.

In view of the above quantum corrections to the HYM equation it is clear that the stability concept relevant for finding solutions to (2.41) likewise has to be modified. As with Π-stability for B-type D-branes [35], in the $E_8 \times E_8$ heterotic string this new notion of stability would correct the tree-level concept of μ-stability.[5]

If we were not dealing with the zero-slope equation (2.41), but instead allowed for some unspecified term const. $\times$ vol. id on the righthand side, the situation would be very similar to the perturbative deformation of the HYM equation as encountered in the context of Gieseker stability [36]. More precisely, we would like to conjecture that this more general, complete loop and non-perturbative corrected HYM equation,

$$\begin{aligned} \star_6 \Bigg[J \wedge J \wedge F_i^{ab} - \frac{\ell_s^4}{4(2\pi)^2} e^{2\phi_{10}} F_i^{ab} \wedge \left(\mathrm{tr}_{E_{8i}}(F_i \wedge F_i) - \frac{1}{2}\mathrm{tr}(R \wedge R) \right) + \\ \ell_s^4 e^{2\phi_{10}} \sum_a N_a \left(\frac{1}{2} \mp \lambda_a \right)^2 F_i^{ab} \wedge \overline{\gamma}_a \Bigg] + (\text{n.p. terms}) = \text{const.} \times \text{vol. id}^{ab}, \end{aligned} \tag{2.42}$$

[5] Π-stability is meant to be the correct notion of stability for B-type D-branes in the limit $g_s = 0$ and to all orders in α'. By S-duality one is tempted to introduce the corresponding stability for heterotic bundles in the limit $g_s \to \infty$, $\alpha' \to 0$ with $\alpha' g_s = const.$

has a solution if the bundle is stable with respect to a corrected slope $\Lambda(\mathcal{F}) = \arg Z(\mathcal{F})$ with the central charge

$$Z(\mathcal{F}) = \frac{1}{2\pi g_s \ell_s^6} \mathrm{Tr} \int_X e^J \left(1 + 2\pi i\alpha' g_s \mathcal{F}\right) \Bigg[1 - \frac{\ell_s^4 g_s^2}{2} \bigg(\frac{1}{4(2\pi)^2} \Big(\mathrm{tr}_{E_{8i}}(F_i \wedge F_i) - \frac{1}{2}\mathrm{tr}(R \wedge R) \Big) - \sum_a N_a \left(\frac{1}{2} \mp \lambda_a \right)^2 \overline{\gamma}_a \bigg) \Bigg] + \text{ (n.p. terms)}. \tag{2.43}$$

We call such a bundle Λ-stable and, neglecting the unknown non-perturbative corrections in (2.43), we call it λ-stable. The reasoning behind this statement is that the tree-level part on the lefthand side of (2.42) can be tuned to dominate arbitrarily over the quantum corrections by choosing the expansion parameter g_s correspondingly small. For more information we refer to [37]. This is no longer possible as soon as we insist that, a forteriori, (2.41) is satisfied, which induces in addition $\Lambda(V) = 0$. After all, we are now cancelling the tree-level and the higher order parts against each other. A more refined analysis of the general quantum corrected stability concept is therefore desirable.

Luckily, for $\mathrm{SU}(N)$ bundles and the particular type of $\mathrm{U}(N)$ bundles treated in this paper a simplification occurs since we will be interested in special bundles defining a heterotic compactification with gauge group flipped $\mathrm{SU}(5) \times \mathrm{U}(1)_X$ and $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)_Y$. As has been shown in [14], the same conditions on the bundles rendering the $\mathrm{U}(1)_X$ and $\mathrm{U}(1)_Y$ massless imply that

$$\lambda(V) = \mu(V) = 0. \tag{2.44}$$

Clearly, this also holds trivially for all $\mathrm{SU}(N)$ bundles. Whenever (2.44) applies, the above arguments imply that λ-stability guarantees the existence of a solution to (2.41). Moreover, for a μ-stable bundle V, we can immediately conclude in this case that it is also λ-stable for sufficiently small string coupling g_s, as for the finite number of subsheaves we can tune g_s such that

$$\lambda(\mathcal{F}) = \mu(\mathcal{F}) + O(g_s^2) < \mu(V) = \lambda(V). \tag{2.45}$$

This is the reason why it is safe for us to work with μ-stable $\mathrm{U}(N)$ bundles, about which much more is known.

We now collect the conditions for μ-stability of our extension bundles V as defined in (2.34). Since V_1 and V_2 are both constructed via irreducible spectral covers, they are guaranteed to be $\mu-$stable with respect to a suitable polarisation, as reviewed in section 2.2. A necessary condition for the extension (2.34) to yield again a *stable* vector bundle is clearly that it be non-split and that $\mu(V_1) < \mu(V) = 0$. Otherwise V_1 would be a subbundle of V with slope not smaller than that of V. It was stated in [28] that this condition is also sufficient. To our knowledge, no proof of this assertion, upon which various models in the literature rely, has been given.[6]. Appendix A contains a detailed proof of this statement.

[6]This was true until the very recent preprint [38], which appeared after our independent analysis on this point had been completed.

More precisely we will show there that

$$\text{V is } \mu\text{−stable w.r.t. } J \Longleftrightarrow H^1(X, V_1 \otimes V_2^\vee) \neq 0 \ \text{ and } \ \mu(V_1) < \mu(V). \tag{2.46}$$

The condition (2.46) can be read as a constraint on the Kähler form J of the manifold X and has to be satisfied inside the Kähler cone such that also V_1 and V_2 are simultaneously stable with respect to it.

3. Computation of cohomology classes

Let us now come to the main technical section of this paper, where we partly review and partly newly derive the mathematical formalism for the computation of the relevant vector bundle cohomology classes. The computation of $H^i(X,V)$ has already been described in very much detail in [27]. Here, we instead compute the more general classes $H^i(X, V_a \otimes V_b)$ and show, using the Leray spectral sequence, that they are localised on the curve $C_a \cap C_b$. The cohomology classes $H^i(X, \bigwedge^2 V)$ were also covered in [27], but our more physically inspired approach gives a deviating result, which however is consistent with the Riemann-Roch-Hirzebruch theorem on the support curve. We also provide the computation of the cohomology classes $H^i(X, \mathbf{S}^2 V)$.[7]

An important tool for the computation of the cohomology of vector bundles on elliptic fibrations is the Leray spectral sequence. More generally, for any fibration $\pi : X \to B$, the Leray spectral sequence relates the cohomology of any bundle V on X to the cohomology of certain sheaves on the base B. These sheaves are called higher direct image sheaves $R^i\pi_* V$ and are defined by

$$R^i\pi_* V(U) = H^i(\pi^{-1}(U), V|_{\pi^{-1}(U)}) \tag{3.1}$$

for any open set $U \subset B$. In particular, observe that for any point $b \in B$

$$R^i\pi_* V|_b = H^i(f_b, V|_{f_b}), \tag{3.2}$$

that is, the higher image sheaf captures the cohomology of V along the fibers f_b of π. In the case of an elliptic fibration, only $R^0\pi_*$ and $R^1\pi_*$ are non-zero and the Leray sequence degenerates to the long exact sequence

$$\begin{array}{c} 0 \longrightarrow H^1(B, \pi_\star V) \longrightarrow H^1(X, V) \longrightarrow H^0(B, R^1\pi_\star V) \longrightarrow \\ \longrightarrow H^2(B, \pi_\star V) \longrightarrow H^2(X, V) \longrightarrow H^1(B, R^1\pi_\star V) \longrightarrow 0 \end{array} \tag{3.3}$$

together with

$$H^0(X,V) = H^0(B, \pi_\star V), \quad H^3(X,V) = H^2(B, R^1\pi_\star V). \tag{3.4}$$

[7] We thank Stefano Guerra for pointing out to us that his upcoming work [39] analyses related questions.

In addition, Serre duality on the one-dimensional fiber implies relative duality, i.e.

$$\left(R^1\pi_\star V\right)^\vee = \pi_\star(V^\vee \otimes K_X \otimes \pi^\star K_B^\vee). \tag{3.5}$$

Another useful relation is the projection formula

$$R^q\pi_\star(V \otimes \pi^\star \mathcal{F}) = R^q\pi_\star(V) \otimes \mathcal{F}, \tag{3.6}$$

for any vector bundle $\mathcal{F}$ on B. To obtain information about the Chern classes of the higher image sheaves one can use the Grothendieck-Riemann-Roch theorem

$$\pi_\star\Big(\mathrm{ch}(V)\,\mathrm{Td}(X)\Big) = \mathrm{ch}(\pi_! \mathrm{V})\,\mathrm{Td}(B) \tag{3.7}$$

with

$$\pi_! V = \sum_{i=1}^{d} (-1)^i\, R^i\pi_* V \tag{3.8}$$

for fiber dimension d. In (3.7) the push-forward $\pi_\star$ of a form is defined as integration over the fiber. The Todd classes are defined in terms of Chern classes as

$$\mathrm{Td}(X) = 1 + \frac{c_1(X)}{2} + \frac{c_2(X) + c_1^2(X)}{12} + \frac{c_1(X)\,c_2(X)}{24} + \dots. \tag{3.9}$$

and simplify considerably for Calabi-Yau manifolds with $c_1(X) = 0$.

3.1 Cohomology classes $H^i(X, \mathcal{L})$

In order to compute the cohomology classes of vector bundles V on X, we need to know how to compute the cohomology classes of any line bundle $\mathcal{L}$ on X. Henceforth, X will be a generic elliptic fibration over a complex two dimensional surface B with zero section σ. Then any line bundle on X will be of the form

$$\mathcal{L} = \mathcal{O}_X(n\sigma) \otimes \pi^\star L \tag{3.10}$$

for some line bundle L on B. Applying the projection formula gives

$$\pi_\star\mathcal{L} = \pi_\star\mathcal{O}_X(n\sigma) \otimes L, \qquad R^1\pi_\star\mathcal{L} = R^1\pi_\star\mathcal{O}_X(n\sigma) \otimes L. \tag{3.11}$$

The higher image sheaves of $\mathcal{O}_X(n\sigma)$ for any n are given by [27],

$$\begin{aligned}
\pi_*(\mathcal{O}_X(n\sigma)) &= \begin{cases} \mathcal{O}_B \oplus \mathcal{O}_B(-2c_1(B)) \oplus \ldots \oplus \mathcal{O}_B(-nc_1(B)) & n \geqslant 0 \\ 0 & n < 0 \end{cases} \\
R^1\pi_*(\mathcal{O}_X(n\sigma)) &= \begin{cases} 0 & n > 0 \\ \mathcal{O}_B((-n-1)c_1(B)) \oplus \ldots & \\ \ldots \oplus \mathcal{O}_B(c_1(B)) \oplus \mathcal{O}_B(-c_1(B)) & n \leqslant 0. \end{cases}
\end{aligned} \tag{3.12}$$

Therefore, in order to apply the Leray spectral sequence, one merely has to determine the cohomology classes $H^i(B, L)$ of general line bundles over the base B. In our case B is a del-Pezzo surface dP_r and we relegate our derivation of $H^i(dP_r, L)$ to appendix B.

3.2 Cohomology classes $H^i(X, V_a \otimes V_b)$

In this section we will show how to compute the cohomology of $V_a \otimes V_b$, where both V_a and V_b are vector bundles on X which admit an irreducible spectral cover.

These cohomology classes are necessary to compute the cohomology of the vector bundles constructed via extensions as described in section 2.4. In addition, they provide a natural setup for computing $H^i(X, \bigwedge^2 V)$ and $H^i(X, \mathbf{S}^2 V)$ for vector bundles V given by the spectral cover construction. Note that the cohomology groups $H^i(X, V_a \otimes V_b) = \mathrm{Ext}^i_X(V_a^\vee, V_b)$ also appear naturally in the SO(32) heterotic and the S-dual Type I string where they count matter fields in bifundamental representations of an $\mathrm{U}(N_a)\times\mathrm{U}(N_b)$ gauge group [13, 33, 40].

To begin with, it is useful to review the results of [27] for the computation of the cohomology of a vector bundle V with vanishing first Chern class given by an irreducible spectral cover C and a globally defined line bundle $\mathcal{N}$. In oder to use the Leray sequence for $\pi : X \to B$ one must find $\pi_* V$ and $R^1\pi_* V$. Recall from section 2.2 that the restriction of V to a generic fiber f_b for $b \in B$ is given by

$$V|_{f_b} = \oplus_{i=1}^{rk(V)} \mathcal{O}_{f_b}(Q_i - p), \tag{3.13}$$

where p and the points Q_i denote the intersection of the fiber f_b with σ and the spectral cover respectively. All of these points are disjoint under our assumption of an irreducible spectral cover. Therefore, $\pi_* V|_b = H^0(f_b, V|_{f_b})$ vanishes for a generic fiber. In addition, for any vector bundle V, $\pi_* V$ is torsion free and hence we find that $\pi_* V$ vanishes identically. It also follows from these considerations that $R^1\pi_* V$ is a sheaf on B supported on the curve $c = C \cap \sigma$ where we identify $B \cong \sigma$. It was shown in [27] that

$$R^1\pi_* V = \mathcal{N} \otimes K_B|_c. \tag{3.14}$$

Applying these results to the Leray spectral sequence determines

$$H^0(X, V) = H^3(X, V) = 0, \;\; H^i(X, V) = H^{i-1}(c, \mathcal{N} \otimes K_B|_c), \; i = 1, 2. \tag{3.15}$$

In particular, it follows that

$$-\chi(X, V) = \chi(c, \mathcal{N} \otimes K_B), \tag{3.16}$$

a result which can be easily checked with the help of the numerical expressions for $c_3(V)$, $\mathcal{N}$ and C of section 2.2.

Consider now two vector bundles V_a and V_b with structure groups $\mathrm{U}(n_a)$ and $\mathrm{U}(n_b)$ given by the spectral cover construction. The two irreducible spectral covers C_a and C_b are in the linear system

$$C_a \in |n_a\, \sigma + \pi^* \eta_a|, \qquad C_b \in |n_b\, \sigma + \pi^* \eta_b| \tag{3.17}$$

and the spectral line bundles $\mathcal{N}_a$ and $\mathcal{N}_b$ are defined as in (2.24). Note that the case $V_a = \mathcal{O}_X$ is included by choosing $C_a = \sigma$ and $\mathcal{N}_a = \mathcal{O}_X$. It follows from the discussion above that the basic strategy to compute the cohomology of $V_a \otimes V_b$ is to compute its spectral data. In particular, we need to find $H^i(c_{ab}, \mathcal{N}_{ab} \otimes K_B|_{c_{ab}})$, $i = 1, 2$ where $\mathcal{N}_{ab}$ is the spectral line bundle, or more generally, a rank one sheaf corresponding to $V_a \otimes V_b$ and c_{ab} is the intersection of the spectral cover of $V_a \otimes V_b$ with the zero section.

Before we attempt to compute $\mathcal{N}_{ab}|_{c_{ab}}$, it is instructive to consider the F-theoretic realization. There the chiral matter is defined by $\mathrm{Ext}^i_X(V_a^\vee, V_b)$, which is expected to be localized on the intersection of the spectral cover for $V_a^\vee$ and V_b, i.e. on the intersection of the two stacks of D7-branes. It follows from (3.13) that the spectral cover for $V_a^\vee$ is τC_a, with τ being the involution defined at the end of section 2.1. Generically, $\tau C_a \cap C_b$ is a smooth curve, denoted by D in the sequel, whose cohomology class is

$$\begin{aligned}[D] \equiv [\tau C_a \cap C_b] &= \pi^* \left(-n_a n_b\, c_1(B) + n_a\, \eta_b + n_b \eta_a\right) \sigma + (\eta_a \eta_b)\, F \\ &= \pi^*[c_{ab}]\, \sigma + a_F\, F. \end{aligned} \tag{3.18}$$

By c_{ab} we denote the projection of the curve on the base. It can be shown using the techniques of [27] that the so-defined class $[c_{ab}]$ is indeed the class of the intersection of the spectral cover of $V_a \otimes V_b$ with the zero section σ.

To find the points $b \in B$ which are contained in c_{ab} consider the restriction of $V_a \otimes V_b$ to a generic fiber f_b

$$V_a \otimes V_b|_{f_b} = V_a|_{f_b} \otimes V_b|_{f_b} = \oplus_{i=1}^{rk(V_a)} \mathcal{O}_{f_b}(Q_i^a - p) \otimes \oplus_{j=1}^{rk(V_b)} \mathcal{O}_{f_b}(Q_j^b - p). \tag{3.19}$$

That is, the intersection points of the spectral cover of $V_a \otimes V_b$ with the fibre f_b are given by the set $\{Q_i^a + Q_j^b\}_{ij}$ and the points $b \in c_{ab}$ are the subset thereof defined by Q_i^a and Q_j^b in the fiber f_b such that

$$Q_i^a + Q_j^b = 0.$$

Here we take addition in the group law of f_b. D and c_{ab} are related via the surjective map

$$\pi_D : \tau C_a \cap C_b \to c_{ab},$$

which is generically one-to-one. As noted above, D is generically smooth, while it is far from obvious that c_{ab} is.

It follows from this discussion that $\pi_*(V_a \otimes V_b)$ vanishes identically. Using the Leray spectral sequence and Serre duality this implies that

$$H^0(X, V_a \otimes V_b) = H^3(X, V_a \otimes V_b) = 0.$$

In addition, $R^1\pi_*(V_a \otimes V_b)$ has support along the curve c_{ab} and is given by $R^1\pi_*(V_a \otimes V_b) = \mathcal{N}_{ab} \otimes K_B|_{c_{ab}}$.

To derive an expression for $\mathcal{N}_{ab}|_{c_{ab}}$, recall from section 2.2 that

$$V|_B = \pi_{C*}\mathcal{N}. \tag{3.20}$$

Hence, at a generic point $b \in B$, we find

$$V|_b = H^0(\pi_C^{-1}(b), \mathcal{N}|_{\pi_C^{-1}(b)}) = \oplus_i (\mathcal{N}|_{Q_i}). \tag{3.21}$$

That implies for the tensor product $V_a \otimes V_b$

$$V_a \otimes V_b|_b = \oplus_{ij}(\mathcal{N}_{ab}|_{Q_i^a+Q_j^b}) = \oplus_{ij}(\mathcal{N}_a|_{Q_i^a} \otimes \mathcal{N}_b|_{Q_j^b}). \tag{3.22}$$

The first and second equality follow from the application of formula (3.21) to $V_a \otimes V_b$ and to V_a and V_b, respectively. Hence we see that the fiber of $\mathcal{N}_{ab}$ at $Q_i + Q_j$ is the tensor product of the fiber of $\mathcal{N}_a$ at Q_i^a with the fiber of $\mathcal{N}_b$ at Q_j^b.

Let us assume that $b \in c_{ab}$. Following the discussion above, this implies that there are two points Q_i^a and Q_i^b obeying $Q_i^b = -Q_j^a$. Replacing $\mathcal{N}_a$ with $\tau^*\mathcal{N}_a$ we find

$$\mathcal{N}_{ab}|_{Q_i^a+Q_j^b} = \tau^*\mathcal{N}_a|_{-Q_i^a} \otimes \mathcal{N}_b|_{Q_j^b}. \tag{3.23}$$

This description is certainly correct for a generic point $b \in c_{ab}$. Therefore, a natural conjecture for the spectral rank one sheaf of $V_a \otimes V_b$ restricted to c_{ab} is

$$\mathcal{N}_{ab}|_{c_{ab}} = \pi_*(\tau^*\mathcal{N}_a \otimes \mathcal{N}_b|_{\tau C_a \cap C_b}). \tag{3.24}$$

In particular, using the finiteness of π_D, we find for the cohomology of $V_a \otimes V_b$

$$\begin{aligned} H^1(X, V_a \otimes V_b) &= H^0(\tau C_a \cap C_b,\, \tau^*\mathcal{N}_a \otimes \mathcal{N}_b \otimes K_B), \\ H^2(X, V_a \otimes V_b) &= H^1(\tau C_a \cap C_b,\, \tau^*\mathcal{N}_a \otimes \mathcal{N}_b \otimes K_B). \end{aligned} \tag{3.25}$$

Observe that for simply connected Calabi-Yau threefolds the Picard group is discrete. Therefore, spectral line bundles $\mathcal{N}_a$ with $\tau^*c_1(\mathcal{N}_a) = c_1(\mathcal{N}_a)$ satisfy $\tau^*\mathcal{N}_a = \mathcal{N}_a$. Note that the spectral line bundles appearing in our applications are just of this type.

In the sequel we will give a numerical proof of these results using general properties of the Fourier-Mukai transform. In particular, we will prove that

$$H^i(X, V_a \otimes V_b) = H^{i-1}(D, \mathcal{L}),\ \ i = 1, 2$$

for some rank one sheaf $\mathcal{L}$ with

$$c_1(\mathcal{L}) = c_1(\mathcal{N}_a \otimes \mathcal{N}_b \otimes K_B))|_D.$$

To begin with, note that the projection formula allows us to write

$$V_a \otimes V_b = \pi_{1\star}\left(\mathcal{P}_B \otimes \pi_2^\star\mathcal{N}_b \otimes \pi_1^*V_a\right). \tag{3.26}$$

Hence we have the two maps

$$\begin{tikzcd} & \Big(X \times_B C_b,\, \mathcal{P}_B \otimes \pi_2^*\mathcal{N}_b \otimes \pi_1^*V_a\Big) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \Big(X,\, V_a \otimes V_b\Big) & & \Big(C_b,\, \pi_{2*}(\mathcal{P}_B \otimes \pi_1^*V_a) \otimes \mathcal{N}_b\Big) \end{tikzcd} \tag{3.27}$$

Since the map π_1 is finite, i.e. its fiber consists of n_a points, the Leray spectral sequence for π_1 reduces to the following relation for the cohomology classes

$$H^i(X \times_B C_b, \mathcal{P}_B \otimes \pi_2^\star \mathcal{N}_b \otimes \pi_1^* V_a) = H^i(X, V_a \otimes V_b) \quad \text{for } i = \{0,1,2,3\}. \tag{3.28}$$

We will now apply the Leray sequence to the projection $\pi_2 : X \times_B C_a \to C_a$. To compute $\pi_{2*}\left(\mathcal{P}_B \otimes \pi_2^* \mathcal{N}_b \otimes \pi_1^* V_a\right)$ consider its restriction to a point $x \in C_b$

$$\begin{aligned} \pi_{2*}\left(\mathcal{P}_B \otimes \pi_2^* \mathcal{N}_b \otimes \pi_1^* V_a\right)|_x &= H^0((\mathcal{P}_B \otimes \pi_2^* \mathcal{N}_b \otimes \pi_1^* V_a)\,|_{E_{\pi(x)\times x}}) \\ &= H^0(\mathcal{O}_{E_{\pi(x)}}(p-x) \otimes (\oplus_j \mathcal{O}_{E_{\pi(x)}}(Q_i^a - p)). \end{aligned} \tag{3.29}$$

Under the assumption of irreducibility of C_a and C_b this vanishes clearly for generic x. Therefore $\pi_{2*}\left(\mathcal{P}_B \otimes \pi_2^* \mathcal{N}_b \otimes \pi_1^* V_a\right)$ vanishes identically, and we find

$$H^i(X \times_B C_a, \mathcal{P}_B \otimes \pi_2^\star \mathcal{N}_b \otimes \pi_1^* V_a) = H^{i-1}(C_b, R^1\pi_{2*}\left(\mathcal{P}_B \otimes \pi_2^* \mathcal{N}_b \otimes \pi_1^* V_a\right)),\ i = 1,2.$$

Combining this result with (3.28) gives

$$H^i(X, V_a \otimes V_b) = H^{i-1}(C_b, R^1\pi_{2*}\left(\mathcal{P}_B \otimes \pi_2^* \mathcal{N}_b \otimes \pi_1^* V_a\right)),\ i = 1,2. \tag{3.30}$$

Note that it follows also from considerations above that $R^1\pi_{2*}\left(\mathcal{P}_B \otimes \pi_2^* \mathcal{N}_b \otimes \pi_1^* V_a\right)$ actually has support only on $\tau C_a \cap C_b$. Hence we define a rank one sheaf on D

$$i_*\mathcal{L} = R^1\pi_{2*}\left(\mathcal{P}_B \otimes \pi_2^* \mathcal{N}_b \otimes \pi_1^* V_a\right),$$

where $i : D \to C_b$ denotes the inclusion map. Using Grothendieck-Riemann-Roch theorem for π_2 we can compute the Chern classes of $i_*\mathcal{L}$. At zero order we find

$$\begin{aligned} \mathrm{ch}_0(R^1\pi_{2*}\left(\mathcal{P}_B \otimes \pi_2^* \mathcal{N}_b \otimes \pi_1^* V_a\right)) &= \pi_{2\star}\left[c_1(W) + \frac{n_b}{2} c_1(X \times_B C_a)\right] \\ &= \pi_{2\star}\left[\Delta - \sigma_1\right] = 0. \end{aligned} \tag{3.31}$$

This was expected, since $R^1\pi_{2*}\left(\mathcal{P}_B \otimes \pi_2^* \mathcal{N}_b \otimes \pi_1^* V_a\right)$ is supported only on the curve D. Similarly, at first order we get

$$c_1(i_*\mathcal{L}) = [C_b \cdot C_a], \tag{3.32}$$

in agreement with $[\tau C_a] = [C_a]$ and $[D] = [C_b \cdot C_a]$. For the second Chern class we obtain

$$\begin{aligned} \mathrm{ch}_2(i_*\mathcal{L}) = &-\lambda_a(-n_a\,\sigma_2 + \pi^*\eta_a - n_a\pi^* c_1(B)) \cdot C_a \cdot C_b \\ &-\lambda_b(-n_b\,\sigma_2 + \pi^*\eta_b - n_b\pi^* c_1(B)) \cdot C_a \cdot C_b \\ &+\left(\frac{1}{n_a} c_1(\zeta_a) + \frac{1}{n_b} c_1(\zeta_b)\right) \cdot C_a \cdot C_b + \frac{1}{2}\, C_a \cdot C_a \cdot C_b - pts. \end{aligned} \tag{3.33}$$

Here the number of points $pts \not\subset C_a \cap C_b$ is given by $pts = (\sigma_2 + c_1(B)) \cdot c_1(B) \cdot C_a = \eta_a \cdot c_1(B) \geq 0$.

The interpretation of this result is as follows: First, the additional class of points appearing in (3.33), as observed in [32], reflects the fact that there exist point-like singularities

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in $X \times_B C_a$. This happens when the discriminant locus meets the branch locus of $C \to B$. These can be blown up leading to changes in the Chern classes such that for the simplest choice of bundle resolution this extra term disappears [32].

We are left with computing the Chern classes of $\mathcal{L}$. Using Grothendieck-Riemann-Roch for the map $i : D \to C_b$, one finds that $\mathcal{L}$ has rank one and that

$$c_1(\mathcal{L}) = \mathrm{ch}_2(i_* \mathcal{L}) + \frac{1}{2} C_a \cdot C_a \cdot C_b. \tag{3.34}$$

As can easily be verified, this implies that

$$c_1(\mathcal{L}) = c_1(\mathcal{N}_a \otimes \mathcal{N}_b \otimes \pi^* K_B)|_D \tag{3.35}$$

thus proving our claim.

To summarize:

$$\boxed{\begin{array}{l}\text{The non-vanishing cohomology classes of the tensor product of two bundles defined}\\ \text{via the spectral cover method can be computed from the cohomology classes of a}\\ \text{certain line bundle on the intersection curve of the two spectral surfaces:}\\[1em] \qquad H^{i+1}(X, V_a \otimes V_b) = H^i(\tau C_a \cap C_b,\, \tau^* \mathcal{N}_a \otimes \mathcal{N}_b \otimes K_B), \quad \text{for } i = 0, 1. \quad (3.36)\end{array}}$$

Consistently, the direct Riemann-Roch-Hirzebruch theorem on the support curve yields the correct Euler characteristic of the bundle $V_a \otimes V_b$ on X,

$$-\chi(X, V_a \otimes V_b) = c_1(\tau^* \mathcal{N}_a \otimes \mathcal{N}_b \otimes K_B)_{C_a \cap C_b} - \frac{1}{2}(C_a + C_b) \cdot C_a \cdot C_b. \tag{3.37}$$

The computation of $H^i(X, V_a \otimes V_b)$ is therefore reduced to the computation of the cohomology of a line bundle over the curve $\tau C_a \cap C_b$ which is the restriction of a line bundle defined on X. The standard procedure in such situations is to invoke a series of Koszul sequences relating the cohomology of the restriction $\mathcal{N}_a \otimes \mathcal{N}_b \otimes K_B$ to that of line bundles on X. The Koszul sequences which do the job for us are displayed in appendix C. Also, the cohomology of line bundles of X is easy to compute in view of section 3.1 and with the help of the results of appendix B.

3.3 Cohomology classes $H^i(X, \bigwedge^2 V)$ and $H^i(X, \mathbf{S}^2 V)$

In this section we compute the cohomology of $\bigwedge^2 V$ and $\mathbf{S}^2 V$ for the case that V is a vector bundle of rank r defined by the spectral cover construction. Since our result differs from the one in [27], we present our derivation in some detail.

To begin with, recall that generally

$$V \otimes V = \left[\bigwedge^2 V\right] \oplus \left[\mathbf{S}^2 V\right]. \tag{3.38}$$

Using the results of the previous section for $V = V_a = V_b$, we can immediately conclude

$$H^{i+1}(X, V \otimes V) = H^i(\tau C \cap C,\, \tau^* \mathcal{N} \otimes \mathcal{N} \otimes K_B), \quad \text{for } i = 0, 1. \tag{3.39}$$

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In the sequel we will again assume that indeed $\tau^*\mathcal{N} = \mathcal{N}$. The zero and third order cohomology groups vanish and the righthand side of (3.39) can be computed using the Koszul sequences in appendix C.

To proceed, we will again use the F-theory respectively Type IIB orientifold intuition. In the orientifold limit of the dual F-theory, the orientifold projection is simply $\Omega I_2(-1)^{F_L}$, where I_2 denotes the holomorphic involution of the fiber T^2. For a stack of N D7-branes wrapping a four-cycle C and carrying a $\mathrm{U}(N)$ gauge group, matter fields transforming in the symmetric and anti-symmetric representations of the gauge group are localized on the intersection of C with the image of C under the holomorphic involution I_2. On the heterotic side the orientifold projection maps precisely to the involution τ discussed at the end of section 2.1. Therefore the matter is localized on the curve

$$D = \tau C \cap C. \tag{3.40}$$

Note that D is invariant under τ. To study the curve D, we consider the fiberwise decomposition of (3.38) for a generic fiber f_b

$$\begin{aligned}(\oplus_i \mathcal{O}_{f_b}(Q_i - p)) \otimes (\oplus_j \mathcal{O}_{f_b}(Q_j - p)) &= (\oplus_{i<j} \mathcal{O}_{f_b}(Q_i + Q_j - 2p)) \\ &\quad \oplus (\oplus_{i \leqslant j} \mathcal{O}_{f_b}(Q_i + Q_j - 2p))\,.\end{aligned} \tag{3.41}$$

We recall from the previous section that the condition on the eigenvalues Q_i of V on a fiber f_b to be in D are

$$Q_j = -Q_i. \tag{3.42}$$

Let us assume $i = j$. Then we find $2Q_i = 0$. This is the intersection of C with the zero section σ and the intersection of C with the triple section σ_t describing points of order two on elliptic fiber. Let us assume that $i \neq j$. Then $Q_i = -Q_j$ implies $Q_j = -Q_i$, hence this fiber contains two points of D. We conclude that D generically consists of three components

$$D = D \cap \sigma + D \cap \sigma_t + D' = C \cap \sigma + C \cap \sigma_t + D'. \tag{3.43}$$

However, the fixed point locus $C \cap \sigma_\tau = C \cap \sigma + C \cap \sigma_t$ intersects D' in

$$R = (C - \sigma_\tau) \cdot C \cdot \sigma_\tau \tag{3.44}$$

points. It follows that for the line bundle $L^2 = \mathcal{N}^2 \otimes K_B$ we have the exact sequence

$$0 \to L^2 \otimes \mathcal{O}(-R)|_{C\cap\sigma_\tau} \to L^2|_{\tau C \cap C} \to L^2|_{D'} \to 0 \tag{3.45}$$

implying the corresponding long exact sequence in cohomology. We now have to split each appearing cohomology group into its τ symmetric and anti-symmetric component. Clearly, the fixed point locus $C \cap \sigma_\tau$ contributes entirely to the cohomology of $\mathbf{S}^2 V$ since we have just identified it with the points $i = j$ appearing in (3.41). Thus

$$H^i_+(C \cap \sigma_\tau, L^2) = H^i(C \cap \sigma_\tau, L^2), \quad H^i_-(C \cap \sigma_\tau, L^2) = 0. \tag{3.46}$$

This is consistent with the orientifold dual, where the fixed point locus of the involution only contributes to anti-symmetric matter and therefore to $H^i(X, \mathbf{S}^2V)$. Therefore, we can conclude that

$$H^{i+1}(X, \bigwedge^2 V) = H^i_-(D', L^2) \tag{3.47}$$

and that $H^{i+1}(X, \mathbf{S}^2V)$ must be determined from the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^0(C \cap \sigma_\tau, L^2 \otimes \mathcal{O}(-R)) \arrow[r] & H^0_+(\tau C \cap C, L^2) \arrow[r] & H^0_+(D', L^2) \arrow[dlll, rounded corners] \\ & H^1(C \cap \sigma_\tau, L^2 \otimes \mathcal{O}(-R)) \arrow[r] & H^1_+(\tau C \cap C, L^2) \arrow[r] & H^1_+(D', L^2) \arrow[r] & 0 \end{tikzcd} \tag{3.48}$$

What remains is to determine $H^i_\pm(D', L^2)$.

To obtain a numerical tool for the computation of this splitting recall from the previous section that

$$H^i(X, \bigwedge^2 V) = H^{i-1}(c_{\wedge^2 V}, \mathcal{N}_{\wedge^2 V} \otimes K_B|_{c_{\wedge^2 V}}),\ i = 1, 2, \tag{3.49}$$

where $c_{\wedge^2 V}$ denotes the intersection of the spectral cover of $\bigwedge^2 V$ with σ and $\mathcal{N}_{\wedge^2 V}$ its spectral rank one sheaf. It is important to realize that the surjective map $\pi_{D'} : D' \to c_{\wedge^2 V}$, which is generically two-to-one, factors through

$$\begin{tikzcd} D' \arrow[r, "\pi_{D'}"] \arrow[d, "m"'] & c_{\wedge^2 V} \\ D'/\tau \arrow[ur, "n"] & \end{tikzcd} \tag{3.50}$$

Here D'/τ is the normalization of $c_{\wedge^2 V}$. The canonical bundle of D' has degree $C(C - \sigma_\tau)(2C - \sigma_\tau)$ and is related to the canonical bundle of D'/τ by

$$m^* K_{D'/\tau} = K_{D'} \otimes \mathcal{O}_{D'}(-R), \tag{3.51}$$

where R is the ramification divisor (3.44). In particular,

$$c_1(K_{D'/\tau})|_{D'/\tau} = \frac{1}{2}(C(C - \sigma_\tau)(2C - \sigma_\tau) - R). \tag{3.52}$$

Applying m_* to L^2, we obtain a rank two vector bundle on D'/τ which splits into a sum of line bundles

$$m_* L^2 = L_i \oplus L_a. \tag{3.53}$$

The sections of L_i and L_a are invariant and anti-invariant under τ, respectively. In particular,

$$m^* L_i = L^2 \tag{3.54}$$

and

$$H^i(D', L^2) = H^i(D', m^* L_i). \tag{3.55}$$

Therefore, in oder to compute the anti-invariant part of the cohomology of L^2 we have to compute $H^i(D'/\tau, L_a)$. Clearly, at generic points $n_* L_a = \mathcal{N}_{c_{\wedge^2 V}} \otimes K_B$, which we expect to hold everywhere. Using GRR for the map m, we find that

$$c_1(L_a)|_{D'/\tau} = c_1\left(\mathcal{N} \otimes K_B^{1/2}\right)|_{D'} + R/2. \tag{3.56}$$

Note that, since $\pi^* K_B|_{D'}$ is invariant under τ, it is the pull-back of a bundle $K_B^{1/2}$ on D'/τ.

Let us summarize our final formulas for the cohomology groups of the anti-symmetric and symmetric product bundles.

- $H^i(X, \bigwedge^2 V)$

> The non-vanishing cohomology groups of the bundle $\bigwedge^2 V$ can be computed from the cohomology groups of a line bundle L_a on the quotient D'/τ by
>
> $$H^{i+1}(X, \textstyle\bigwedge^2 V) = H^i_-(D', L^2) = H^i(D'/\tau, L_a), \quad \text{for } i = 0, 1.$$
>
> with the first Chern class of L_a given by
>
> $$c_1(L_a)|_{D'/\tau} = c_1\left(\mathcal{N} \otimes K_B^{1/2}\right)|_{D'} + R/2. \tag{3.57}$$

Applying the RRH theorem to this line bundle we find

$$-\chi(X, \textstyle\bigwedge^2 V) = c_1\left(\mathcal{N} \otimes K_B^{1/2}\right)|_{D'} - \dfrac{1}{2} C \cdot C \cdot (C - \sigma_\tau), \tag{3.58}$$

an important consistency check of our computation.

- $H^i(X, \mathbf{S}^2 V)$

> The non-vanishing cohomology groups of the bundle $\mathbf{S}^2 V$,
>
> $$H^{i+1}(X, \mathbf{S}^2 V) = H^i_+(\tau C \cap C, L^2), \tag{3.59}$$
>
> can be computed from the sequence (3.48) with
>
> $$H^i_+(D', L^2) = H^i(D'/\tau, L_i), \quad \text{for } i = 0, 1,$$
>
> and the first Chern class of L_i is given by
>
> $$c_1(L_i)|_{D'/\tau} = c_1\left(\mathcal{N} \otimes K_B^{1/2}\right)|_{D'}. \tag{3.60}$$

In particular, this implies together with the sequence (3.48) that $-\chi(X, \mathbf{S}^2 V) = \chi(C \cap \sigma_\tau, L^2 \otimes \mathcal{O}(-R)) + \chi_+(D', L^2)$, and again the RRH theorem consistently gives

$$-\chi(X, \mathbf{S}^2 V) = c_1\left(\mathcal{N} \otimes K_B^{1/2}\right)|_{\tau C \cap C} - \dfrac{1}{2} C \cdot C \cdot (C + \sigma_\tau). \tag{3.61}$$

4. Heterotic flipped SU(5) GUT models

Having presented the mathematical framework for the computation of the complete massless spectrum of heterotic string compactification with $U(N)$ bundles, we are now in a position to apply these techniques to heterotic model building. After briefly summarizing the way flipped SU(5) vacua were obtained in [14] we present a new fully consistent three-generation string vacuum. For this phenomenologically promising model, we exemplify the methods developed in the first part of this paper and compute the complete massless spectrum.

4.1 $SU(4) \times U(1)$ bundles

We consider a bundle with structure group $SU(4) \times U(1)$ on a Calabi-Yau X including cases with $\pi_1(X) = 0$. Such types of construction were considered in [41] before and further details of this particular one can be found in [12, 14].[8]

More precisely, our starting point is the direct sum

$$W_1 = V_1 \oplus L^{-1} \quad \text{with } c_1(V_1) = c_1(L), \ \ \mathrm{rank}(V) = 4, \tag{4.1}$$

where V_1 is $U(4)$ vector bundle, L is a complex line bundle and the structure group of W_1 is $G_1 = SU(4) \times U(1)$ due to the constraint $c_1(V_1) = c_1(L)$. G_1 can now be embedded into an SU(5) subgroup of the first E_8 such that its commutant in E_8 is $SU(5) \times U(1)_1$. For the details of embeddings of this type we refer to [12, 37]. The decomposition of the adjoint **248** of E_8,

$$\mathbf{248} \overset{SU(4)\times SU(5)\times U(1)_1}{\longrightarrow} \left\{ \begin{array}{c} (\mathbf{15},\mathbf{1})_0 \\ (\mathbf{1},\mathbf{1})_0 + (\mathbf{1},\mathbf{10})_{-4} + (\mathbf{1},\overline{\mathbf{10}})_4 + (\mathbf{1},\mathbf{24})_0 \\ (\mathbf{4},\mathbf{1})_5 + (\mathbf{4},\overline{\mathbf{5}})_{-3} + (\mathbf{4},\mathbf{10})_1 \\ (\overline{\mathbf{4}},\mathbf{1})_{-5} + (\overline{\mathbf{4}},\mathbf{5})_3 + (\overline{\mathbf{4}},\overline{\mathbf{10}})_{-1} \\ (\mathbf{6},\mathbf{5})_{-2} + (\mathbf{6},\overline{\mathbf{5}})_2 \end{array} \right\}, \tag{4.2}$$

reveals that the spectrum is precisely that of flipped $SU(5) \times U(1)_X$ [18] provided we guarantee that the abelian gauge group remains massless in the process of Green-Schwarz type anomaly cancellation.

In [14] it was proposed to embed a second line bundle into the other E_8 such that a linear combination of the two observable U(1)'s remains massless. It turns out, however, that in order to construct models with precisely the Standard Model matter content and no further non-chiral matter, it is more convenient to invoke in the hidden E_8 a slightly more sophisticated structure than the one detailed in [14]. Namely, we can consider the second simplest embedding

$$U(2) \times U(1) \subset E_8 \to E_6 \times U(1)_2 \tag{4.3}$$

[8] For different aspects of this and related constructions see [42]. Recent investigations of heterotic K3 compactifications with line bundles are performed in [43, 44]. Some previous results on heterotic $U(N)$ bundles in six and five-dimensional compactifications appear in [45, 46] and [47], respectively.

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$E_6 \times \mathrm{U}(1)_2$	cohomology
$\mathbf{1}_3$	$V_2 \otimes L$
$\mathbf{27}_1$	V_2
$\mathbf{27}_{-2}$	L^{-1}

Table 2: Massless spectrum of $H = E_6 \times \mathrm{U}(1)_2$ models.

inducing the decomposition

$$\mathbf{248} \xrightarrow{\mathrm{SU}(2)\times E_6 \times \mathrm{U}(1)_2} \left\{ \begin{array}{c} (\mathbf{1},\mathbf{78})_0 \\ (\mathbf{1},\mathbf{1})_0 + (\mathbf{3},\mathbf{1})_0 + (\mathbf{2},\mathbf{1})_3 + (\mathbf{2},\mathbf{1})_{-3} \\ (\mathbf{2},\mathbf{27})_1 + (\mathbf{1},\mathbf{27})_{-2} \\ (\mathbf{2},\overline{\mathbf{27}})_{-1} + (\mathbf{1},\overline{\mathbf{27}})_2 \end{array} \right\}. \tag{4.4}$$

The bundle we embed into the second E_8 is of the form

$$W_2 = V_2 \oplus L^{-1}, \qquad c_1(V_2) = c_1(L), \tag{4.5}$$

where we stress that the line bundle L is the same as the one appearing in (4.1). The resulting chiral spectrum in the second E_8 is counted by the cohomology groups listed in table 2.

The unitary vector bundles V_1, V_2 and the complex line bundle L are subject to a number of constraints to guarantee that the model constitutes a well-defined string vacuum with the desired spectrum. The non-trivial Bianchi identity for the three-form field strength ensuring anomaly cancellation translates into the following cohomological constraint on the second Chern classes of the vector bundles and the tangent bundle of the Calabi-Yau,

$$\mathrm{ch}_2(V_1) + \mathrm{ch}_2(V_2) + c_1^2(L) - \sum_a N_a \overline{\gamma}_a = -c_2(X). \tag{4.6}$$

Here we allowed for the presence of stacks of N_a space-time-filling five-branes wrapping the holomorphic two-cycles Γ_a dual to the four-form $\overline{\gamma}_a$ on X.

According to the reasoning detailed in [14], the combination

$$\mathrm{U}(1)_f = -\frac{1}{2}\left(\mathrm{U}(1)_1 - \frac{5}{3}\,\mathrm{U}(1)_2\right) \tag{4.7}$$

remains massless if the following conditions are satisfied

$$\begin{aligned} \int_X c_1(L) \wedge c_2(V_1) &= 0, \\ \int_X c_1(L) \wedge c_2(V_2) &= 0, \\ \int_{\Gamma_a} c_1(L) &= 0 \quad \text{for all M5 branes.} \end{aligned} \tag{4.8}$$

$\mathrm{SU}(5) \times \mathrm{U}(1)_X \times E_6$	Cohomology	χ	SM part.
$(\mathbf{10}, \mathbf{1})_{\frac{1}{2}}$	$H^i(V_1)$	g	$(q_L, d_R^c, \nu_R^c) + [H_{10} + \overline{H}_{10}]$
$(\mathbf{10}, \mathbf{1})_{-2}$	$H^i(L^{-1})$	0	$-$
$(\overline{\mathbf{5}}, \mathbf{1})_{-\frac{3}{2}}$	$H^i(V_1 \otimes L^{-1})$	g	(u_R^c, l_L)
$(\overline{\mathbf{5}}, \mathbf{1})_1$	$H^i(\bigwedge^2 V_1)$	0	$[(h_3, h_2) + (\overline{h}_3, \overline{h}_2)]$
$(\mathbf{1}, \mathbf{1})_{\frac{5}{2}}$	$H^i(V_1 \otimes L) + H^i(V_2^\vee \otimes L^{-1})$	g	e_R^c
$(\mathbf{1}, \mathbf{27})_{\frac{5}{6}}$	$H^i(V_2)$	0	$-$
$(\mathbf{1}, \mathbf{27})_{-\frac{5}{3}}$	$H^i(L^{-1})$	0	$-$

Table 3: Massless spectrum of $H = \mathrm{SU}(5) \times \mathrm{U}(1)_X$ models with hidden E_6 symmetry; $g = \frac{1}{2}\int_X c_3(V_1)$.

In this case, the one-loop correction (2.43) to the slope vanishes and we are in the fortunate situation of (2.44), i.e. that μ-stability of V_1 and V_2, together with the DUY condition (2.39), guarantees supersymmetry in the perturbative regime.

Chiral right-handed electrons from the second E_8 with non-MSSM Yukawa couplings are absent if in addition

$$\int_X c_1^3(L) = 0. \tag{4.9}$$

In this case one obtains no chiral matter at all resulting from the hidden E_8. The complete spectrum can be found in table 3.

Whereas the net number of chiral generations is simply given by $\frac{1}{2}\int_X c_3(V_1)$ once the constraints (4.6), (4.8), (4.9) are satisfied, the vector-like matter is described by the cohomology groups listed in table 3. Our task is therefore to find stable vector $\mathrm{U}(4)$ and $\mathrm{U}(2)$ bundles V_1 and V_2 as well as a line bundle L subject to the constraints discussed such that

- $H^*(X, V_1) = (0, 1, 4, 0)$ for precisley one pair of GUT Higgs and 3 generations of (q_L, d_R^c, ν_R^c),
- $H^*(X, V_1 \otimes L^{-1}) = (0, 0, 3, 0)$ for 3 generations of (u_R^c, l_L),
- $H^*(X, V_1 \otimes L) = (0, 0, 3, 0)$ for 3 generations of e_R^c and
- $H^*(X, L^{-1}) = H^*(X, V_2) = (0, 0, 0, 0) = H^*(X, V_2 \otimes L)$ in order to avoid all kinds of non-chiral exotic matter.

The number of Higgs pairs is then determined by $H^*(X, \bigwedge^2 V_1)$.

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4.2 A three-generation model from extensions

We now provide an example of the flipped SU(5) framework whose spectrum comes remarkably close to the Standard Model. Our Calabi-Yau manifold X is elliptically fibered over the basis B, which we take to be $\mathrm{d}P_4$.

Let us start with the visible E_8, into which we embed the direct sum $W_1 = V_1 \oplus L^{-1}$ of a stable $\mathrm{U}(4)$ bundle V_1 and the line bundle L with $c_1(V_1) = c_1(L)$. For L we choose the pull-back of a line bundle on dP_4 with first Chern class

$$c_1(L) = -E_1 + E_4. \tag{4.10}$$

Since L is the pull-back of a line bundle on the base space, clearly

$$\int_X c_1(L)^3 = 0 \tag{4.11}$$

and we therefore have no further contributions to the right-handed electrons.

V_1 is constructed as the extension of two stable $\mathrm{U}(2)$ bundles V_a and V_b,

$$0 \to V_a \to V_1 \to V_b \to 0, \tag{4.12}$$

where V_a and V_b are obtained via the spectral cover construction. Concretely, the defining data is in the notation of section 2

$$\begin{aligned} &\lambda_a = 0,\ c_1(\zeta)_a = l - 2E_2, &&\eta_a = 12l - 5E_1 - 5E_2 - 3E_3 - 5E_4, \\ &\lambda_b = 0,\ c_1(\zeta)_b = -l - E_1 + 2E_2 + E_4, &&\eta_b = 10l - 4E_1 - E_2 - 3E_3 - 4E_4. \end{aligned}$$

Clearly $c_1(V_1) = c_1(V_a) + c_1(V_b) = -E_1 + E_4 = c_1(L)$, as required.

In the hidden E_8 we embed $W_2 = V_2 \oplus L^{-1}$, where the $\mathrm{U}(2)$ bundle V_2 is given by the data

$$\lambda_2 = 0, \quad c_1(\zeta)_2 = -E_1 + E_4, \quad \eta_2 = 7l - 2E_1 - 3E_2 - 3E_3 - 2E_4, \tag{4.13}$$

again satisfying $c_1(V_2) = c_1(L)$. One can check that each of the bundles V_a, V_b and V_2 are stable in that they satisfy the corresponding criteria described in section 2.2.

Proof of stability. We now prove stability of the bundle V_1 defined via the extension (4.12). As discussed in section 2.5 and in appendix A, this amounts to showing that the extension is non-split and that $\mu(V_a) < \mu(V_1) = 0$ for a Kähler class inside the Kähler cone. Starting with this latter constraint, we parameterise the Kähler form J on X as $J = \ell_s^2(r_\sigma\, \sigma + \pi^*(r_0 l + \sum_{i=1}^4 r_i E_i))$. Note that r_σ measures the area of the fibre. The values for r_σ, r_0 and r_i have to be such that J lies inside the Kähler cone. For the numerical constraints following from this requirement we refer e.g. to appendix A of [33]. One may check that they are satisfied for the choice

$$0 < r_\sigma < 4\rho, \quad r_0 = 6\rho, \quad r_1 = -2\rho, \quad r_2 = -3\rho, \quad r_3 = -2\rho, \quad r_4 = -2\rho, \tag{4.14}$$

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where $\rho > 0$. Note that for this choice, the DUY-condition (2.39) for V_1 and V_2 is fulfilled and therefore $\mu(V_{1,2}) = 0$. Stability of each of the bundles V_a, V_b and V_2 requires furthermore that $r_\sigma < \epsilon$ for some (in general unknown) critical value of ϵ. With the help of the intersection form on the basis we readily compute that

$$\int_X J \wedge J \wedge c_1(V_a) = \ell_s^4 \left(-r_\sigma^2 + 2r_\sigma(r_0 + 2r_2)\right). \tag{4.15}$$

Restricting oneself for simplicity to the parameter space in (4.14) one concludes that $\mu(V_a) < 0$ translates into

$$r_\sigma^2 > 0, \tag{4.16}$$

which is always true. Most importantly, stability of V_1 does therefore not set a lower bound on r_σ so that we can indeed take it smaller than any critical ϵ required by the spectral cover construction.

The second part of the stability condition on V_1 requires the computation of $H^1(X, V_a \otimes V_b^\vee)$. According to equation (3.25), $H^i(X, V_a \otimes V_b^\vee) = H^{i-1}(C_a \cap C_b, \mathcal{L}|_{C_a \cap C_b})$ for $i = 1, 2$, where $\mathcal{L} = \mathcal{N}_{V_a} \otimes \mathcal{N}_{V_b^\vee} \otimes K_B$. As discussed in appendix C, we have to invoke a series of three Koszul sequences in which four line bundles on X appear: $\mathcal{L}, \mathcal{L} \otimes \mathcal{O}(-\mathcal{C}_a), \mathcal{L} \otimes \mathcal{O}(-\mathcal{C}_b), \mathcal{L} \otimes \mathcal{O}(-\mathcal{C}_a - \mathcal{C}_b)$. Their cohomology groups are easily determined once we know their first Chern classes. From the definition of the spectral line bundle (2.24) and the concrete bundle data we find

$$\begin{aligned} c_1(\mathcal{N}_{V_a}) &= \sigma + \pi_C^* \left(\frac{1}{2}(\eta_a + c_1(B)) + \frac{1}{n} c_1(\zeta)_a\right), \\ c_1(\mathcal{N}_{V_b^\vee}) &= \sigma + \pi_C^* \left(\frac{1}{2}(\eta_b + c_1(B)) - \frac{1}{n} c_1(\zeta)_b\right). \end{aligned} \tag{4.17}$$

Thus,

$$\begin{aligned} c_1(\mathcal{L}) &= 2\sigma + \pi_C^*(12l - 4E_1 - 5E_2 - 3E_3 - 5E_4), \\ c_1(\mathcal{L} \otimes \mathcal{O}(-\mathcal{C}_a)) &= \pi_C^*(E_1), \\ c_1(\mathcal{L} \otimes \mathcal{O}(-\mathcal{C}_b)) &= \pi_C^*(2l - 4E_2 - E_4), \\ c_1(\mathcal{L} \otimes \mathcal{O}(-\mathcal{C}_a - \mathcal{C}_b)) &= -2\sigma + \pi_C^*(-10l + 5E_1 + E_2 + 3E_3 + 4E_4) \end{aligned} \tag{4.18}$$

with Hodge numbers

$$\begin{aligned} H^*(X, \mathcal{L}) &= (57, 0, 0, 0), \\ H^*(X, \mathcal{L} \otimes \mathcal{O}(-\mathcal{C}_a)) &= (1, 0, 0, 0), \\ H^*(X, \mathcal{L} \otimes \mathcal{O}(-\mathcal{C}_b)) &= (0, 5, 6, 0), \\ H^*(X, \mathcal{L} \otimes \mathcal{O}(-\mathcal{C}_a - \mathcal{C}_b)) &= (0, 0, 0, 39). \end{aligned} \tag{4.19}$$

In all we find

$$H^*(X, V_a \otimes V_b^\vee) = (0, 61, 45, 0) \tag{4.20}$$

and therefore the extension is non-split. This completes the proof of stability of V_1.

Checking the consistency conditions. In section 2.2, we listed the Chern characters for spectral cover bundles (see (2.26)). The result for the two vector bundles V_a and V_b in this example is

$$\begin{aligned} \mathrm{ch}_1(V_a) &= l - 2E_2, \\ \mathrm{ch}_2(V_a) &= \sigma\pi^*(-12l + 5E_1 + 5E_2 + 3E_3 + 5E_4) + \frac{13}{2}F, \\ \mathrm{ch}_3(V_a) &= -1, \end{aligned} \tag{4.21}$$

$$\begin{aligned} \mathrm{ch}_1(V_b) &= -l - E_1 + 2E_2 + E_4, \\ \mathrm{ch}_2(V_b) &= \sigma\pi^*(-10l + 4E_1 + 1E_2 + 3E_3 + 4E_4) + \frac{11}{2}F, \\ \mathrm{ch}_3(V_b) &= 4. \end{aligned} \tag{4.22}$$

V_1 being the extension of V_b by V_a its total Chern character is the sum of the total Chern characters of V_a and V_b. Thus,

$$\begin{aligned} \mathrm{ch}_1(V_1) &= -E_1 + E_4, \\ \mathrm{ch}_2(V_1) &= \sigma\pi^*(-22l + 9E_1 + 6E_2 + 6E_3 + 9E_4) + 12F, \\ \mathrm{ch}_3(V_1) &= 3. \end{aligned} \tag{4.23}$$

The Chern classes are then

$$\begin{aligned} c_1(V_1) &= -E_1 + E_4, \\ c_2(V_1) &= \sigma\pi^*(22l - 9E_1 - 6E_2 - 6E_3 - 9E_4) - 13F, \\ c_3(V_1) &= 6. \end{aligned} \tag{4.24}$$

From the second Chern class, one can easily read off that the first line of the masslessness conditions (4.8) holds.

For the U(2)-bundle in the hidden E_8, V_2, the Chern characters come out to be

$$\begin{aligned} \mathrm{ch}_1(V_2) &= -E_1 + E_4, \\ \mathrm{ch}_2(V_2) &= \sigma\pi^*(-7l + 2E_1 + 3E_2 + 3E_3 + 2E_4) + F, \\ \mathrm{ch}_3(V_2) &= 0. \end{aligned} \tag{4.25}$$

To satisfy the tadpole condition (4.6), the Poincaré dual four-form of the two-cycles, the five-branes are wrapping must be:

$$\begin{aligned} \sum_a N_a \overline{\gamma_a} &= \mathrm{ch}_2(V_1) + \mathrm{ch}_2(V_2) + c_1(L)^2 + c_2(X) \\ &= \sigma\pi^*(7l - E_1 - 3E_2 - 3E_3 - E_4) + 73F. \end{aligned} \tag{4.26}$$

This can be decomposed in a sum of positive multiples of irreducible cycles, for example:

With this decomposition, it is easy to see that the second line in equation (4.8)

$$\int_{\Gamma_a} c_1(L) = \int_X c_1(L) \wedge \overline{\gamma_a} = 0 \tag{4.27}$$

indeed holds for all components.

a	N_a	$\overline{\gamma_a}$
1	1	$\sigma\pi^*(l - E_1 - E_4)$
2	6	$\sigma\pi^*(l - E_2 - E_3)$
3	3	$\sigma\pi^*(E_2)$
4	3	$\sigma\pi^*(E_3)$
5	73	F

Computation of the massless spectrum. As mentioned in section 2.4, we can calculate the cohomology groups $H^*(X, V_1)$ by the long exact sequence in cohomology, induced by (4.12):

$$\begin{aligned} 0 \to H^0(X, V_a) \to H^0(X, V_1) \to H^0(X, V_b) \to \\ \to H^1(X, V_a) \to H^1(X, V_1) \to H^1(X, V_b) \to \dots \end{aligned} \tag{4.28}$$

For the cohomology groups $H^*(X, V_a)$ and $H^*(X, V_b)$, one can use again the method described in section 3.2 by considering the tensor product with the trivial vector bundle $\mathcal{O}_X$ respectively. The results are

$$\begin{aligned} H^*(X, V_a) &= H^*(X, V_a \otimes \mathcal{O}_X) = (0, 1, 1, 0), \\ H^*(X, V_b) &= H^*(X, V_b \otimes \mathcal{O}_X) = (0, 0, 3, 0) \end{aligned} \tag{4.29}$$

and therefore

$$H^*(X, V_1) = (0, 1, 4, 0). \tag{4.30}$$

The exact sequence from the extension (4.12) remains exact upon tensoring every element with a line bundle. Thus, we can calculate the cohomology groups $H^*(X, V_1 \otimes L)$ and $H^*(X, V_1 \otimes L^{-1})$ by the long exact sequence in cohomology, induced by the tensored short exact sequence. We find

$$\begin{aligned} H^*(X, V_a \otimes L) &= (0, 0, 0, 0), \\ H^*(X, V_a \otimes L^{-1}) &= (0, 0, 0, 0), \\ H^*(X, V_b \otimes L) &= (0, 0, 3, 0), \\ H^*(X, V_b \otimes L^{-1}) &= (0, 0, 3, 0), \end{aligned} \tag{4.31}$$

yielding

$$\begin{aligned} H^*(X, V_1 \otimes L) &= (0, 0, 3, 0), \\ H^*(X, V_1 \otimes L^{-1}) &= (0, 0, 3, 0). \end{aligned} \tag{4.32}$$

For the computation of the cohomology groups $H^*(X,V_1)$, we use that the short exact sequence (2.34) induces the following set of exact sequences

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \bigwedge^2 V_a & \to & Q_1 & \to & V_a\otimes V_b & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \bigwedge^2 V_a & \to & \bigwedge^2 V_1 & \to & Q_2 & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \bigwedge^2 V_b & & \bigwedge^2 V_b & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{4.33}$$

Since V_a and V_b are bundles of rank 2, their anti-symmetric product is actually a line bundle and its cohomology can be computed using the method described in section 3.1. The result is

$$\begin{aligned}
H^*(X,\textstyle\bigwedge^2 V_a) &= (0,0,1,0),\\
H^*(X,\textstyle\bigwedge^2 V_b) &= (0,2,1,0),\\
H^*(X,V_a\otimes V_b) &= (0,53-\mathrm{rk}\,f,53-\mathrm{rk}\,f,0),
\end{aligned} \tag{4.34}$$

where f is the map $f:H^1(X,\mathcal{L}\otimes\mathcal{O}(-\mathcal{C}_a)|_{\mathcal{C}_b})\to H^1(X,\mathcal{L}|_{\mathcal{C}_b})$ appearing in the Koszul sequences. These spaces are both one-dimensional, so f might in principle have rank 0 or 1. Resolving the various induced long exact sequences in cohomology in (4.33) gives

$$\begin{aligned}
H^*(X,Q_1) &= (0,53-\mathrm{rk}\,f-\mathrm{rk}\,g,54-\mathrm{rk}\,f-\mathrm{rk}\,g,0),\\
H^*(X,Q_2) &= (0,55-\mathrm{rk}\,f-\mathrm{rk}\,h,54-\mathrm{rk}\,f-\mathrm{rk}\,h,0),
\end{aligned} \tag{4.35}$$

where g and h are the maps $g:\ H^1(X,V_a\otimes V_b)\to H^2(X,\wedge^2 V_a)$, $h:\ H^1(X,\wedge^2 V_b)\to H^2(X,V_a\otimes V_b)$. From the dimensions of their image and domain, (4.34), one can read off that their ranks can at most lie in the ranges $[0,1]$ and $[0,2]$ respectively. Using these results, the exact sequence for $\bigwedge^2 V_1$ gives

$$H^*(X,\textstyle\bigwedge^2 V_1) = (0,55-\mathrm{rk}\,f-\mathrm{rk}\,g-\mathrm{rk}\,i,55-\mathrm{rk}\,f-\mathrm{rk}\,g-\mathrm{rk}\,i,0), \tag{4.36}$$

where the rank of $i:\ H^1(X,Q_1)\to H^2(X,V_a\otimes V_b)$ can be in the range $[0,2]$. Thus we have at least $H^*(X,\bigwedge^2 V_1)=(0,51,51,0)$.

In the hidden sector, the cohomology is

$$\begin{aligned}
H^*(X,V_2) &= (0,0,0,0),\\
H^*(X,V_2^\vee\otimes L^{-1}) &= (0,2-\mathrm{rk}\,j,2-\mathrm{rk}\,j,0),
\end{aligned} \tag{4.37}$$

where the rank of $j:H^1(\sigma,\mathcal{N}_{V_2}\otimes K_B\otimes\mathcal{O}(-C)|_\sigma)\to H^1(\sigma,\mathcal{N}_{V_2}\otimes K_B|_\sigma)$ can again lie at most within the range $[0,2]$.

A remark about the actual ranks of the linear maps f,g,h,i,j is in order. Their concrete value depends on the choice of bundle moduli and can therefore vary over the

$\mathrm{SU}(5) \times \mathrm{U}(1)_X \times E_6$	Cohomology	χ	SM part.
$(\mathbf{10},\mathbf{1})_{\frac{1}{2}}$	$(0,1,4,0)$	3	$(q_L, d_R^c, \nu_R^c) + [H_{10} + \overline{H}_{10}]$
$(\mathbf{10},\mathbf{1})_{-2}$	$(0,0,0,0)$	0	$-$
$(\overline{\mathbf{5}},\mathbf{1})_{-\frac{3}{2}}$	$(0,0,3,0)$	3	(u_R^c, l_L)
$(\overline{\mathbf{5}},\mathbf{1})_{1}$	$(0,[51,55],[51,55],0)$	0	$[(h_3,h_2)+(\overline{h}_3,\overline{h}_2)]$
$(\mathbf{1},\mathbf{1})_{\frac{5}{2}}$	$(0,0,3,0)$	3	e_R^c
$(\mathbf{1},\mathbf{27})_{\frac{5}{6}}$	$(0,0,0,0)$	0	$-$
$(\mathbf{1},\mathbf{27})_{-\frac{5}{3}}$	$(0,0,0,0)$	0	$-$

Table 4: Massless spectrum of a flipped SU(5) model with hidden E_6 symmetry.

moduli space. To decide which values they can really take within the naive ranges stated above requires a more thorough analysis as performed, in the context of $\mathrm{SU}(N)$ bundles, in [27]. Since it is of phenomenological relevance, we restrict our attention here to the rank of the map j, which decides about the appearance of possible exotic matter in the form of extra right-handed electrons. A detailed, but straightforward analysis along the lines of [27] reveals that the possible values for $\mathrm{rk}\, j$ are 0 and 2, with 2 being the generic value and 0 corresponding to a specific choice of bundle moduli for V_2. We therefore restrict ourselves to the generic maximal value leading indeed to $H^*(X, V_2^\vee \otimes L^{-1}) = (0,0,0,0)$, as desired. For this generic choice of moduli the number of Higgses is then in the range $[51, 55]$ and can be determined in a similar manner, though we do not perform this analysis here.

To conclude, we list again the total spectrum of our example in table 4.

5. Conclusions

In this paper we have provided the technical tools for the computation of the complete massless spectrum of heterotic string compactifications invoking vector bundles with $\mathrm{U}(N)$ structure groups. Our main results are both of purely mathematical interest and lead, from the physical point of view, to the construction of new quasi-realistic heterotic string compactifications.

Taking the one-loop corrections to the Donaldson-Uhlenbeck-Yau equation derived in [12] seriously, we have proposed a new notion of stability for the loop and non-perturbatively corrected Hermitian Yang-Mills equation. It is the analogue of the concept of Π-stability of B-type D-branes [35] for vector bundles in the $E_8 \times E_8$ heterotic string. While, in the context of importance to us in this publication, this modified stability concept reduces to the familiar one of μ-stability, it would be interesting to investigate the implications of Λ-stability both from the mathematical point of view and with respect to applications in string model building.

In the technical main part of this article we have extended the results of [27] concerning the computation of cohomology groups for vector bundles defined via the spectral cover method. In particular, we have provided the expressions for $H^i(X, V_a \otimes V_b)$, $H^i(X, \bigwedge^2 V)$

and $H^i(X, \mathbf{S}^2V)$, where for the latter two our results differ significantly from the ones obtained in [27]. In all these cases the cohomology can be computed from certain line bundles living on the intersection curves of the two spectral cover surfaces involved. Therefore, eventually the technical computation of the massless spectrum boils down to the determination of the cohomologies of line bundles on certain curves. For U(4) bundles defined via non-split extensions of two U(2) bundles, we have provided an explicit proof for their μ-stability.

In the remaining, more physically oriented part of this paper we have applied all these techniques to the construction of stringy flipped SU(5) models as proposed in [14], where the masslessness of the $\mathrm{U}(1)_X$ introduced additional constraints on the $\mathrm{SU}(4)\times\mathrm{U}(1)$ bundle involved. Defining the U(4) bundle via an extension of two U(2) bundles, we have found what we believe is the first fully consistent, supersymmetric flipped SU(5) string model with just the MSSM matter spectrum, i.e. without any additional vector-like matter. Moreover, this model exhibits precisely one vector-like pair of the desired GUT Higgs fields in the antisymmetric representation of SU(5) allowing for field theoretic GUT symmetry breaking down to the Standard Model gauge group. The only major shortcoming is the appearance of a large number of electroweak Higgs fields. The common philosophy how to deal with unwanted vector-like pairs would be to carefully analyse their mass matrix and determine whether they can acquire sufficiently large masses as to decouple from the effective low-energy theory (for recent examples in the heterotic literature see e.g. [48 − 51]). We leave such an analysis for future work, but hasten to stress that the extra Higgs pairs are a consequence of the very specific geometric background and the types of vector bundles employed and may be avoidable in different setups.

As discussed in [14], for the type of flipped SU(5) vacua studied in this article there are no obvious selection rules forbidding any of the observed Yukawa couplings, whereas potentially problematic dimension four, five and six operators inducing unacceptable proton decay are absent. We consider this latter issue as a clear phenomenological advantage which is known to distinguish flipped SU(5) from the Georgi-Glashow GUT scenario. Further phenomenological studies would involve the actual computation of the relevant interaction terms including the ones involving the GUT Higgs and which are required for the field theoretic symmetry breaking.

As an alternative to this type of GUT model building, it would be interesting to apply the methods of this paper to the construction of string vacua directly with MSSM gauge group, along the lines of [14]. These vacua are defined by embedding a vector bundle of structure group $\mathrm{SU}(5)\times\mathrm{U}(1)$ into E_8, yielding $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)_Y$ in four dimensions. We plan to come back to these questions in the future.

What we find most interesting in the light of recent discussions concerning the gauge sector of the four-dimensional string landscape (e.g. [52 − 57, 51]) is the fact that the model presented in this article is just one example of a much larger class of heterotic vacua which, as we recall, are defined on general simply-connected Calabi-Yau manifolds. In particular, they do not rely on highly non-trivial properties of the fundamental group of the internal space or on full solvability of the underlying CFT. We are quite confident that by extending the analysis of this paper to more generic backgrounds and vector bundles, models with

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just the MSSM spectrum can be found.

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A. Proof of stability for rank four bundles V by two generic rank two bundles V_1, V_2

Let X be a generic elliptically fibered Calabi-Yau manifold as considered in section 2.1. In particular we assume that X admits exactly one section.[9] Consider a vector bundle V defined by the short exact sequence

$$0 \to V_1 \to V \to V_2 \to 0, \tag{A.1}$$

where V_i are $\mathrm{U}(N)$ or $\mathrm{SU}(N)$ bundles over X that correspond to two different generic irreducible spectral covers. This implies that the restriction of V_i to a generic fiber is isomorphic to the sum of degree zero line bundles which are mutually different and that V_i is stable with respect to any ample class of the form $J = J_X + n\pi^* J_B$ for sufficiently large n. Here J_X and J_B denote ample classes on X and B, respectively.

In the sequel, we restrict ourselves to the case that both V_i are of rank 2, as in section 4. Consider a bundle V that corresponds to a non-trivial element in $Ext^1(V_2, V_1)$ and satisfies $\mu(V_1) < \mu(V) < \mu(V_2)$ with $\mu(V) = 0$. We will show that under these assumptions V is a stable bundle with respect to any ample class of the form $J = J_X + n\pi^* J_B$ for sufficiently large n. In particular, we have to show that all torsion free sheaves $\mathcal{N}$ of rank smaller than four which admit an injective map

$$\mathcal{N} \to V$$

obey $\mu(\mathcal{N}) < 0$. Note that, as discussed for example in [58], it is sufficient to show this statement for all vector bundles of rank smaller than four.

Since the restriction of V to a generic fiber is by construction isomorphic to the sum of mutually different degree zero line bundles, the degree of all subbundles of V along the generic fiber is smaller than or equal to zero. As can be seen by straightforward computation, for subbundles of degree smaller than zero along the generic fiber, large n is sufficient to make their slope negative, hence they cannot destabilize V [15].

Therefore we have to consider only subbundles of degree zero along the generic fiber. The fact that the spectral cover of V is the union of the two irreducible spectral covers of V_1 and V_2 implies that these subbundles are of rank two.

[9] Otherwise, we assume that $\wedge^2 V_2$ restricted to a generic fiber is trivial.

Instead of checking for destabilising rank two subbundles of V, we can check for destabilising sub line bundles of $\bigwedge^2 V$.[10] For this purpose we make use of the fact that $\bigwedge^2 V$ fits into the exact sequence

$$\begin{tikzcd} & & \bigwedge^2 V_2 \\ \bigwedge^2 V_1 \arrow[r] & \bigwedge^2 V \arrow[r] & Q \arrow[u] \\ & & V_1 \otimes V_2 \arrow[u] \end{tikzcd} \tag{A.2}$$

It follows that the subbundles $\mathcal{N}$ of $\bigwedge^2 V$ are either subbundles of $\bigwedge^2 V_1$ or subbundles of Q lifting to $\bigwedge^2 V$. In the first case these bundles cannot be destabilising since

$$\mu(\mathcal{N}) \leq \mu(\bigwedge^2 V_1) < \mu(\bigwedge^2 V).$$

Note that the first inequality is due to fact that the rank of $\bigwedge^2 V_1$ is one and hence the cokernel of $\mathcal{N} \to \bigwedge^2 V_1$ is a torsion sheaf. It remains to show that subbundles of Q with non-negative slope do not lift to $\bigwedge^2 V$.

Every line bundle which is a subbundle of Q must either be a subbundle of $V_1 \otimes V_2$ or a subbundle of $\bigwedge^2 V_2$ lifting to Q. However, for generic spectral covers of V_1 and V_2, $V_1 \otimes V_2$ itself corresponds to an irreducible spectral cover and therefore has no subbundles of rank one and degree zero along the fiber, as discussed above.

Turning to the second possibility, we note that it follows from our assumptions that $\bigwedge^2 V_2 = \pi^* L_2$ for some line bundle L_2 on B. Consider subbundles of $\pi^* L_2$ of degree zero along the fiber. They are of the form $\pi^* D$ for some line bundle D on B. In order for them to destabilize $\bigwedge^2 V$ they have to lift to $\bigwedge^2 V$, hence they have to lift to Q.

We will show that this is impossible. As a standard matter of fact, every diagram

$$\begin{tikzcd} V_1 \otimes V_2 \arrow[r] & Q \arrow[r] & \pi^* L_2 \\ & & \pi^* D \arrow[u] \end{tikzcd} \tag{A.3}$$

can be completed to

$$\begin{tikzcd} & & \pi^* F \\ V_1 \otimes V_2 \arrow[r] & Q \arrow[r] & \pi^* L_2 \arrow[u] \\ V_1 \otimes V_2 \arrow[r] \arrow[u] & Q' \arrow[r] \arrow[u] & \pi^* D \arrow[u] \end{tikzcd} \tag{A.4}$$

for some sheaf F with support on a divisor S on B (see e.g. section III of [59]). It is easy to see that in our specific case F is a line bundle on S. In addition $\pi^* D$ lifts to Q if and

[10] This is a special case of the general fact that for a rank r subbundle W_r of a rank m bundle V_m also $\bigwedge^r W_r \subset \bigwedge^r V_m$.

only if Q' corresponds to the trivial extension, i.e. $Q^{'} = 0 \in Ext^1(\pi^* D, V_1 \otimes V_2)$ [59]. We can assume that Q is not the trivial extension. There exists a natural map

$$Ext^1(\pi^* L_2, V_1 \otimes V_2) \rightarrow Ext^1(\pi^* D, V_1 \otimes V_2)$$

and we can complete our proof by showing that this map is an injection.

To do so consider the short exact sequence

$$0 \rightarrow \pi^* D \rightarrow \pi^* L_2 \rightarrow \pi^* F \rightarrow 0 \tag{A.5}$$

in (A.4) inducing the long exact sequence

$$\cdots \rightarrow Ext^1(\pi^* F, V_1 \otimes V_2) \rightarrow Ext^1(\pi^* L_2, V_1 \otimes V_2) \rightarrow Ext^1(\pi^* D, V_1 \otimes V_2) \rightarrow \cdots \tag{A.6}$$

We conclude that a sufficient condition for Q' not being the trivial extension is the vanishing of $Ext^1(\pi^* F, V_1 \otimes V_2)$. Consider

$$\begin{aligned} Ext^1_X(\pi^* F, V_1 \otimes V_2) &= Ext^2_X(V_1 \otimes V_2, \pi^* F)^* && \text{(A.7)} \\ &= H^2(X, \pi^* F \otimes V_1^* \otimes V_2^*)^* && \text{(A.8)} \\ &= H^2(\pi^* S, V_1^* \otimes V_2^* \otimes \pi^* F) && \text{(A.9)} \\ &= H^0(\pi^* S, V_1 \otimes V_2 \otimes \pi^* F^* \otimes K_{\pi^* S}) && \text{(A.10)} \\ &= H^0(S, \pi_*(V_1 \otimes V_2 \otimes K_{\pi^* S}) \otimes F^*) = 0, && \text{(A.11)} \end{aligned}$$

where we use Serre duality on X and on $\pi^* S$. The last equality follows from the fact that $\pi_*(V_1 \otimes V_2) = 0$ for generic spectral cover bundles V_i.

B. Cohomology of line bundles over del-Pezzo surfaces

In order to determine the cohomology classes of line bundles over general del Pezzo surfaces dP_r, $r = 0, \ldots, 8$ we proceed as follows. We will first compute the effect of blowing up just a single point on $\mathbb{P}_2$ and will then argue that the different blow-ups are independent of each other. This leads directly to the general formula for r blown up points.

Blowing up just a single point results in dP_1, which is the same as the Hirzebruch surface F_1. The latter is a $\mathbb{P}_1$ fibration over $\mathbb{P}_1$ and we can therefore apply the Leray spectral sequence for this fibration $dP_1 = F_1 \xrightarrow{\pi} \mathbb{P}_1$. More concretely, consider a line bundle on dP_1 with first Chern class

$$c_1(L) = a\, l + b\, E_1 = a\,(l - E_1) + (b + a)\, E_1, \tag{B.1}$$

where $S = E_1$ and $\mathcal{E} = l - E_1$ are precisely the two $\mathbb{P}_1$s appearing in F_1. The intersection form for these 2-cycles is

$$S \cdot S = -1, \quad S \cdot \mathcal{E} = 1, \quad \mathcal{E} \cdot \mathcal{E} = 0. \tag{B.2}$$

As has been shown in [27] and can be verified by utilizing for instance the Grothendieck-Riemann-Roch theorem the push-forward of a line bundle onto the $\mathbb{P}_1$ described by the divisor S is

$$\begin{aligned} \pi_*(L) &= \mathcal{O}(a\mathcal{E}) \otimes [\mathcal{O} \oplus \mathcal{O}(-\mathcal{E}) \oplus \ldots \oplus \mathcal{O}(-(a+b)\mathcal{E})] \quad \text{for } a+b \geq 0 \\ R^1\,\pi_*(L) &= \mathcal{O}(a\mathcal{E}) \otimes [\mathcal{O}(\mathcal{E}) \oplus \mathcal{O}(2\mathcal{E}) \oplus \ldots \oplus \mathcal{O}(-(a+b+1)\mathcal{E})] \quad \text{for } a+b < 0. \end{aligned} \tag{B.3}$$

Applying now Bott's formula for the cohomology classes of line bundles on $\mathbb{P}_1$ gives the cohomology classes of the push-forward line bundles on $\mathbb{P}_1$,

$$H^0(\mathbb{P}_1, \pi_* L) = \begin{cases} \binom{a+2}{2} & \text{for } a \geq 0,\ b \geq 0 \\ \binom{a+2}{2} - \binom{b}{2} & \text{for } a \geq 0,\ -a \leq b < 0 \\ 0 & \text{else} \end{cases} \tag{B.4}$$

and

$$H^1(\mathbb{P}_1, \pi_* L) = \begin{cases} \binom{b}{2} & \text{for } a \geq 0,\ b \geq 0 \\ -\binom{a+2}{2} + \binom{b}{2} & \text{for } a < 0,\ b > -a \\ 0 & \text{else.} \end{cases} \tag{B.5}$$

Similarly, for the cohomology classes of the first right derived functor we find

$$H^0(\mathbb{P}_1, R^1\,\pi_* L) = \begin{cases} -\binom{a+2}{2} + \binom{b}{2} & \text{for } a \geq 0,\ b < -a \\ \binom{b}{2} & \text{for } a < 0,\ b < 0 \\ 0 & \text{else} \end{cases} \tag{B.6}$$

and

$$H^1(\mathbb{P}_1, R^1\,\pi_* L) = \begin{cases} \binom{a+2}{2} & \text{for } a < 0,\ b < 0 \\ \binom{a+2}{2} - \binom{b}{2} & \text{for } a < 0,\ 0 < b < -a \\ 0 & \text{else.} \end{cases} \tag{B.7}$$

With the help of the Leray spectral sequence it is now straightforward to compute $H^i(dP_1, L)$. Using the above decoupling argument for the different blow-ups the final result for general del-Pezzo surfaces dP_r can be written in the following suggestive form. Consider the general line bundle on dP_r with

$$c_1(L) = a_0\, l + \sum_{i=1}^{\rho} b_i\, E_i + \sum_{j=\rho+1}^{r} c_j\, E_j \text{ with } b_i < 0 \text{ and } c_j \geq 0. \tag{B.8}$$

For $a_0 \geq 0$ define

$$A = \binom{a_0+2}{2} - \sum_{i=1}^{\rho} \binom{b_i}{2}. \tag{B.9}$$

If $A \geq 0$ the cohomology classes of the line bundle are

$$H^*(dP_r, L) = \left(A, \sum_{j=\rho+1}^{r} \binom{c_j}{2}, 0 \right) \tag{B.10}$$

and for $A < 0$ they are

$$H^*(dP_r, L) = \left(0, \sum_{j=\rho+1}^{r} \binom{c_j}{2} - A, 0 \right) . \tag{B.11}$$

Similarly, if $a_0 < 0$ we define

$$A = \binom{a_0 + 2}{2} - \sum_{i=1}^{\rho} \binom{c_j}{2} . \tag{B.12}$$

If $A \geq 0$ the cohomology classes of the line bundle are

$$H^*(dP_r, L) = \left(0, \sum_{i=1}^{\rho} \binom{b_i}{2}, A \right) \tag{B.13}$$

and for $A < 0$ they are

$$H^*(dP_r, L) = \left(0, \sum_{i=1}^{\rho} \binom{b_i}{2} - A, 0 \right) . \tag{B.14}$$

Of course these formulae are consistent with the Riemann-Roch-Hirzebruch formula for the Euler characteristic of these line bundles over dP_r. In addition we have checked that for the toric del-Pezzo surfaces $dP_0, \dots, dP_3$ they are consistent with the cohomology classes derived using toric methods.

C. Koszul sequence for $H^*(X, V_a \otimes V_b)$

As derived in section (3.2), the cohomology groups of the tensor product of two spectral cover bundles V_a and V_a are given by

$$H^i(X, V_a \otimes V_b) = H^{i-1}(C_a \cap C_b, \mathcal{L}|_{C_a \cap C_b}) \ \text{for}\, i = 1, 2, \tag{C.1}$$

where $\mathcal{L} = \mathcal{N}_{V_a} \otimes \mathcal{N}_{V_b} \otimes K_B$. Our task is thus to compute the cohomology $H^*(C_a \cap C_b, \mathcal{L}|_{C_a \cap C_b})$ for a line bundle $\mathcal{L}$ defined on the elliptically fibered three-fold X. This can be accomplished by invoking the Koszul sequence

$$(I) \quad 0 \to \mathcal{L} \otimes \mathcal{O}(-C_a)|_{C_b} \to \mathcal{L}|_{C_b} \to \mathcal{L}|_{C_a \cap C_b} \to 0 \ . \tag{C.2}$$

The point is that each of the first two objects can again be computed from known objects on X via a Koszul sequence of its own,

$$(II) \quad 0 \to \mathcal{L} \otimes \mathcal{O}(-C_b) \to \mathcal{L} \to \mathcal{L}|_{C_b} \to 0 \tag{C.3}$$

and

$$(III) \quad 0 \to \mathcal{L} \otimes \mathcal{O}(-C_a - C_b) \to \mathcal{L} \otimes \mathcal{O}(-C_a) \to \mathcal{L} \otimes \mathcal{O}(-C_a)|_{C_b} \to 0 \ . \tag{C.4}$$

Each of these three short exact sequences induces a long exact sequence in cohomology.

We therefore need as our input data the dimensions of the cohomology groups of the four line bundles on X

$$\begin{aligned} H^*(X, \mathcal{L}) &= H^*(X, \mathcal{N}_{V_a} \otimes \mathcal{N}_{V_b} \otimes K_B), \\ H^*(X, \mathcal{L} \otimes \mathcal{O}(-C_a)) &= H^*(X, \mathcal{N}_{V_a} \otimes \mathcal{N}_{V_b} \otimes K_B \otimes \mathcal{O}(-C_a)), \\ H^*(X, \mathcal{L} \otimes \mathcal{O}(-C_b)) &= H^*(X, \mathcal{N}_{V_a} \otimes \mathcal{N}_{V_b} \otimes K_B \otimes \mathcal{O}(-C_b)), \\ H^*(X, \mathcal{L} \otimes \mathcal{O}(-C_a - C_b)) &= H^*(X, \mathcal{N}_{V_a} \otimes \mathcal{N}_{V_b} \otimes K_B \otimes \mathcal{O}(-C_a - C_b)). \end{aligned} \tag{C.5}$$

These can easily be obtained with the help of the general expressions for the cohomology groups of line bundles on X given in section 3.1 together with appendix B.

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